GLUING OF PERVERSE SHEAVES ON THE BASIC AFFINE SPACE

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with an appendix by R. Bezrukavnikov and A. Polishchuk

The goal of this paper is to present the foundations of the gluing techniques introduced by D. Kazhdan and G. Laumon in [11]. They used this techniques to construct an abelian category $\mathcal{A}$ attached to a split semi-simple group G over a finite field $\mathbb{F}_q$ equipped with actions of G, W (the Weil group) and the Frobenius automorphism. More precisely, the category $\mathcal{A}$ is glued from $|W|$ copies of the category of perverse sheaves on the basic affine space G/U (where U is the unipotent radical of the Borel subgroup of G). Conjecturally the category $\mathcal{A}$ contains information about all "forms" (in the algebro-geometric sense) of principal series representations of $G(\mathbb{F}_q)$. In this paper we explain the Kazhdan—Laumon construction in details and study gluing of categories in a broader context.

The idea of gluing of categories is based on the geometric model of gluing a variety from open pieces. One just has to translate geometric data into the categorical framework by passing to categories of sheaves. Loosely speaking, in order to glue several categories one has to provide functors between them which generate a ring-like structure in the monoidal category of functors. When we want to glue abelian categories we have to impose in addition some exactness conditions on these functors. Then the glued category will also be abelian. The principal difficulty arises when one tries to study homological properties of the glued category. The intuition coming from the geometric model here is misleading: it turns out that even if the original categories have finite cohomological dimension the glued category can have an infinite dimension. The original motivation for the present work was a conjecture made in [11] that the category $\mathcal{A}$ has finite cohomological dimension. This conjecture turned out to be wrong: a counterexample was found by R. Bezrukavnikov (see Appendix). However, for applications to representation theory a weaker version of this conjecture would suffice. Namely, it would be enough to prove that objects of finite cohomological dimension generate the (suitably localized) Grothendieck group of the category $\mathcal{A}$ and of its twisted versions. In this paper we check that this is true (for the category $\mathcal{A}$ itself) in the cases when G is of type A_n, $n \leq 3$, or B_2. Also, we have recorded an important theorem of L. Positselski asserting that the individual Ext-groups in the glued category are finite-dimensional (this was conjectured in [11]).

The functors used by Kazhdan and Laumon for their gluing are certain generalized partial Fourier transforms F_w on the category of sheaves on G/U parametrized by elements w of the Weil group W. An important observation made in [11] is that the functors F_w produce an action of the generalized positive braid monoid B^+ associated with W on the category of sheaves on G/U. In this paper we axiomatize this situation in the data which we call W-*gluing* (in particular, the categories to be glued are numbered by elements of W). By combining this abstract formalism with some geometric considerations we give a proof of Theorem 2.6.1 of [11] (the proof given by the authors is insufficient). We also define "restriction" and "induction" functors between glued categories associated with parabolic subgroups in W. This allows us to deduce an interesting property of simple objects in the category $\mathcal{A}$. Namely, every such object has a *support* which is a subset of W. We prove that the support of a simple object in $\mathcal{A}$ is either the entire set W or a convex subset of W (in the sense of the metric given by reduced decompositions). Another topic discussed in this paper is an analogue of the Kazhdan-Laumon gluing when W is replaced by its "half": a subset of elements $w \in W$ with $\ell(sw) = \ell(w) + 1$ where s is a fixed simple reflection (the set

1991 *Mathematics Subject Classification.* 20C33, 14F20, 14G15.
Key words and phrases. perverse sheaves, gluing, Coxeter group, basic affine space.

of w with $\ell(sw) = \ell(w) - 1$ is "another half"). Applying parabolic induction functors we show that the homological properties of an object in the corresponding glued category are completely defined by its restrictions to the glued categories for proper parabolic subgroups of W

The basic idea of Kazhdan and Laumon was that the category $\mathcal{A}$ "looks" as if it were the category of perverse sheaves on a scheme of finite type. To extend this analogy we introduce a mixed version of the category $\mathcal{A}$ and prove a bound on the weights of Ext-groups between pure objects which is known to hold in the case of perverse sheaves on a scheme – this gives one more reason to believe that the category $\mathcal{A}$ is "similar" to such a category.

As was shown in [6], for regular characters of a finite torus Kazhdan—Laumon gluing gives the same representation as the one defined by Deligne and Lusztig in [10]. It seems that the methods of the present paper can be applied to try to analyze the Kazhdan—Laumon construction in the case of characters that are "not very singular".

Here is the plan of the paper. In section 1 we give basic definitions concerning gluing of abelian categories. In section 2 we compute the Grothendieck group of the glued category (under some mild assumptions). In section 3 we introduce the notion of W-gluing where W is a Coxeter group by axiomatizing the situation of gluing on G/U. Here we prove a general theorem in the spirit of Deligne's paper [9] on *quasi-actions* of a Coxeter group W on a category (by this we mean a collection of functors $(F_w, w \in W)$ and of morphisms $F_w F_{w'} \to F_{ww'}$ that satisfy the natural associativity condition but need not be isomorphisms). Section 4 is devoted to the Fourier transform on a symplectic vector bundle and its restriction to the complement of the zero section. In section 5 we consider gluing on G/U. In particular, we prove Theorem 2.6.1 of [11] which asserts that generalized Fourier transforms define a quasi-action of W on sheaves on G/U. In section 6 we introduce the technical notion of the cubic Hecke algebra $\mathcal{H}^c$ and prove that the action of the generalized braid group B on $K_0(G/U)$ factors through $\mathcal{H}^c$. In section 7 we give explicit construction of adjoint functors between glued categories associated with parabolic subgroups of W. Here we prove the theorem on supports of simple objects in the glued category. Also we construct some canonical complexes of functors on the glued category. In particular, for objects supported on "half" of W, we construct canonical resolutions consisting of objects induced from parabolic subgroups of W. We also prove that for a gluing on "half" of W our gluing procedure coincides with the one considered by Beilinson and Drinfeld in [2]. In section 8 we study Ext-groups in the glued category. Section 9 is devoted to the gluing of mixed perverse sheaves. In section 10 we describe an embedding of the category of representations of the braid group corresponding to a parabolic subgroup $W_J \subset W$ into the category of representations of the braid group B. We also consider an analogue of this picture for quasi-actions of these groups on categories. Finally, in section 11 we introduce the notion of a *good* representation of the generalized braid group B and prove that some representations of B are good. This is applied to constructing quotients of $K_0(\mathcal{A}^{\mathrm{Fr}})$ on which the hypothetical bilinear form induces non-degenerate forms. In Appendix, written jointly with R. Bezrukavnikov, we show that the category $\mathcal{A}$ has infinite cohomological dimension for $G = \mathrm{SL}_3$.

Acknowledgments. I am grateful to D. Kazhdan for introducing me to the subject and for guidance at various stages of work on this project. My understanding of gluing has been clarified a lot due to numerous discussions with A. Braverman and L. Positselski. Moreover, the latter has made an important contribution by proving the theorem on finiteness of dimensions of Ext-groups in the glued category, as well as pointing out an error in my earlier optimistic proof of finiteness of cohomological dimension. I am grateful to G. Laumon and M. Rapoport for communicating to me the proof of commutativity of the diagram (4.2.2) which plays an important role in checking the associativity condition for the morphism from the square of the Fourier transform to the identity functor. This work was partially supported by NSF grant DMS-9700458.

Notation. Let k be a field that is either finite or algebraically closed. For a scheme S of finite type over k and a prime l different from the characteristic of k, we denote by $\mathcal{D}^b_c(S, \overline{\mathbb{Q}}_l)$ the triangulated category of

constructible l-adic sheaves on X defined by Deligne in [7]. We denote by $\mathrm{Perv}(S) \subset \mathcal{D}_c^b(S, \overline{\mathbb{Q}}_l)$ the abelian subcategory of perverse sheaves defined in [3]. For a morphism $f : S \to S'$ between such schemes we denote by f^*, $f_!$ and f_* the corresponding derived functors between categories $\mathcal{D}_c^b(S, \overline{\mathbb{Q}}_l)$ and $\mathcal{D}_c^b(S', \overline{\mathbb{Q}}_l)$. For a Coxeter group (W, S) we denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the corresponding length function, by B the corresponding generalized braid group and by B^+ its positive brais submonoid. We denote by $\tau : W \to B$ the canonical section given by reduced decompositions. To avoid confusion with the notation for the set of cosets we denote the set-theoretic difference by $X - Y$ (instead of the customary $X \setminus Y$).

1. Basic definitions

1.1. Gluing of abelian categories.

Let $(\mathcal{C}_i)$, $i = 1, \dots, n$ be a collection of abelian categories.

Definition 1.1.1. A *(left) gluing data* for $(\mathcal{C}_i)$ is a collection of right-exact (covariant) functors

$$\Phi_{i,j} : \mathcal{C}_j \to \mathcal{C}_i \tag{1.1.1}$$

for all pairs (i, j) such that $\Phi_{i,i} = \mathrm{Id}_{\mathcal{C}_i}$, and a collection of morphisms of functors

$$\nu_{i,j,k} : \Phi_{i,j} \circ \Phi_{j,k} \to \Phi_{i,k} \tag{1.1.2}$$

for all triples (i, j, k) such that $\nu_{i,i,k} = \mathrm{id}$, $\nu_{i,j,j} = \mathrm{id}$ and the following associativity equation holds:

$$\nu_{i,j,l} \circ (\Phi_{i,j}\nu_{j,k,l}) = \nu_{i,k,l} \circ (\nu_{i,j,k}\Phi_{k,l}) \tag{1.1.3}$$

for all quadruples (i, j, k, l).

One defines *right gluing data* similarly by inverting arrows and requiring functors to be left-exact. Henceforth, the term "gluing data" by default will refer to the "left gluing data".

For a gluing data $\Phi = (\Phi_{i,j}; \nu_{i,j,k})$ we define the category $\mathcal{C}(\Phi)$ as follows. The objects of $\mathcal{C}(\Phi)$ are collections $(A_i; \alpha_{ij})$ where A_i is an object of $\mathcal{C}_i$ $(i = 1, \dots, n)$, $\alpha_{ij} : \Phi_{i,j}A_j \to A_i$ is a morphism in $\mathcal{C}_i$ (for every pair (i, j)) such that the following diagram is commutative:

$$\begin{tikzcd}
\Phi_{i,j}\Phi_{j,k}A_k \arrow[r, "\Phi_{i,j}\alpha_{jk}"] \arrow[d, "\nu_{i,j,k}"] & \Phi_{i,j}A_j \arrow[d, "\alpha_{ij}"] \\
\Phi_{i,k}A_k \arrow[r, "\alpha_{ik}"] & A_i
\end{tikzcd} \tag{1.1.4}$$

for every triple (i, j, k).

A morphism $f : (A_i; \alpha_{ij}) \to (A'_i; \alpha'_{ij})$ is a collection of morphisms $f_i : A_i \to A'_i$ such that $f_i \circ \alpha_{ij} = \alpha'_{ij} \circ \Phi_{i,j}(f_j)$ for all (i, j).

Lemma 1.1.2. *The category $\mathcal{C}(\Phi)$ is abelian.*

Proof. For a morphism $f : (A_i; \alpha_{ij}) \to (A'_i, \alpha'_{ij})$ in $\mathcal{C}(\Phi)$ there are natural objects of $\mathcal{C}(\Phi)$ extending the collections $(\ker(f_i))$ and $(\mathrm{coker}(f_i))$, which constitute the kernel and the cokernel of f. Indeed, the composition of the natural morphism

$$\Phi_{i,j}\ker(f_j) \to \Phi_{i,j}A_j \to A_i$$

with $f_i : A_i \to A'_i$ is zero; hence, it factors through a morphism $\Phi_{i,j}\ker(f_j) \to \ker(f_i)$, and we get the structure of an object of $\mathcal{C}(\Phi)$ on $(\ker(f_i))$. Similarly, the composition of the natural morphism

$$\Phi_{i,j}A'_j \to A'_i \to \mathrm{coker}(f_i)$$

with $\Phi_{i,j}f_j : \Phi_{i,j}A_j \to \Phi_{i,j}A'_j$ is zero; hence, it factors through a morphism $\mathrm{coker}(\Phi_{i,j}f_j) \to \mathrm{coker}(f_i)$. However, $\mathrm{coker}(\Phi_{i,j}f_j) \simeq \Phi_{i,j}\,\mathrm{coker}(f_j)$ since the functor $\Phi_{i,j}$ is right-exact. Thus, we get a structure of an object of $\mathcal{C}(\Phi)$ on the collection $(\mathrm{coker}(f_i))$. It follows that both the cokernel of kernel and kernel of

cokernel of f coincide with the natural object extending the collection $(\mathrm{im}(f_i))$, and therefore, $\mathcal{C}(\Phi)$ is abelian. $\square$

Remarks.
1. Dualizing the above construction we obtain the glued category for right gluing data.
2. The more general "gluing" procedure is obtained by considering an abelian category $\mathcal{C}$ with a right-exact functor $\Phi : \mathcal{C} \to \mathcal{C}$ together with morphisms of functors $\mathrm{Id} \to \Phi$ and $\Phi^2 \to \Phi$ such that Φ is a monoid object in the category of functors from $\mathcal{C}$ to itself. In our situation $\mathcal{C} = \oplus_i \mathcal{C}_i$ and Φ has components $\Phi_{i,j}$. Note that if $\mathcal{C}$ is the category of vector spaces over a field k, then any k-algebra A induces a gluing data on $\mathcal{C}$ in the above generalized sense. The functor Φ in this case is tensoring with A and the glued category is just the category of A-modules.

1.2. **Adjunctions.** Let Φ be a left gluing data for $(\mathcal{C}_i)$. The functor $j_k^* : \mathcal{C}(\Phi) \to \mathcal{C}_k : (A_i; \alpha_{ij}) \mapsto A_k$ has the left adjoint $j_{k,!} : \mathcal{C}_k \to \mathcal{C}(\Phi)$ such that $j_k^* \circ j_{k,!} = \mathrm{Id}_{\mathcal{C}_k}$. Namely,

$$j_{k,!}(A) = (\Phi_{i,k}(A); \nu_{i,j,k})$$

where

$$\nu_{i,j,k} : \Phi_{i,j}\Phi_{j,k}(A) \to \Phi_{i,k}(A)$$

is the structural morphism of the gluing data.

The following theorem states that the existence of such adjoint functors essentially characterizes the glued category.

Theorem 1.2.1. *Let $\mathcal{C}_k$, $k = 1, \dots, n$ be a collection of abelian categories, and $\mathcal{C}$ an abelian category equipped with exact functors $j_k^* : \mathcal{C} \to \mathcal{C}_k$ for $k = 1, \dots, n$. Assume that for every k there exists the left adjoint functor $j_{k,!} : \mathcal{C}_k \to \mathcal{C}$ such that $j_k^* \circ j_{k,!} = \mathrm{Id}_{\mathcal{C}_k}$. Assume also that for an object $A \in \mathcal{C}$ the condition $j_k^* A = 0$ for all k implies that $A = 0$. Then $\mathcal{C}$ is equivalent to $\mathcal{C}(\Phi)$ where Φ is the gluing data with $\Phi_{ij} = j_i^* \circ j_{k,!}$.*

The proof is not difficult and we leave it to the reader. A more general statement of this kind is Theorem 2.6 of [5].

Assume that the functor $\Phi_{i,j}$ has the right adjoint $\Psi_{j,i} : \mathcal{C}_i \to \mathcal{C}_j$ for every (i,j). Then the functors $\Psi_{j,i}$ give rise to a right gluing data for $(\mathcal{C}_i)$. Namely, by adjunction we have a natural morphism of functors $\mathrm{Id}_{\mathcal{C}_k} \to \Psi_{k,j} \circ \Psi_{j,i} \circ \Phi_{i,k}$ induced by $\nu_{i,j,k}$, so we can form the composition

$$\lambda_{k,j,i} : \Psi_{k,i} \to \Psi_{k,j} \circ \Psi_{j,i} \circ \Phi_{i,k} \circ \Psi_{k,i} \to \Psi_{k,j} \circ \Psi_{j,i}$$

where $\Phi_{i,k} \circ \Psi_{k,i} \to \mathrm{Id}_{\mathcal{C}_i}$ is the canonical adjunction morphism. One can see that the associativity condition analogous to (1.1.3) holds for $\lambda_{i,j,k}$. On the other hand, the morphism of functors $\nu_{j,i,j} : \Phi_{j,i} \circ \Phi_{i,j} \to \mathrm{Id}_{\mathcal{C}_j}$ by adjunction gives rise to a morphism

$$\mu_{i,j} : \Phi_{i,j} \to \Psi_{i,j} \tag{1.2.1}$$

for every (i,j).

By construction the glued categories $\mathcal{C}(\Phi)$ and $\mathcal{C}(\Psi)$ are canonically equivalent. In particular, by duality we have right adjoint functors $j_{k,*} : \mathcal{C}_k \to \mathcal{C}(\Phi)$ to the restriction functor $j_k^* : \mathcal{C}(\Phi) \to \mathcal{C}_k$:

$$j_{k,*}(A_k) = (\Psi_{i,k}(A_k); \alpha'_{ij})$$

where the morphisms $\alpha'_{ij} : \Phi_{i,j}\Psi_{j,k}A_l \to \Psi_{i,k}A_k$ are deduced by adjunction from the right gluing data $(\Psi_{i,j}, \lambda_{i,j,k})$. One has $j_k^* \circ j_{k,*} = \mathrm{Id}_{\mathcal{C}_k}$.

1.3. **Middle extensions and simple objects.** The morphism (1.2.1) gives rise to a morphism of functors $\mu_k : j_{k,!} \to j_{k,*}$, such that for every object $A \in \mathcal{C}(\Phi)$ the composition of the adjunction morphisms

$$j_{k,!}j_k^*A \to A \to j_{k,*}j_k^*A$$

coincides with $\mu_l(j_k^*A)$. Thus, we can define the middle extension functor $j_{k,!*} : \mathcal{C}_k \to \mathcal{C}(\Phi)$

$$j_{k,!*}(A_k) = \operatorname{im}(j_{k,!}(A) \to j_{k,*}(A)).$$

Lemma 1.3.1. *With the above assumption for every simple object $A \in \mathcal{C}(\Phi)$ and any l the restriction $j_l^*A \in \mathcal{C}_l$ is either simple or zero.*

Proof. Assume that $j_l^*A \neq 0$ and that there exist an exact sequence

$$0 \to B_l \to j_l^*A \to C_l \to 0$$

with non-zero B_l and C_l. Then by adjunction we have non-zero morphisms $f : j_{l,!}(B_l) \to A$ and $g : A \to j_{l,*}(C_l)$ such that $g \circ f = 0$. Since A is simple, f should be surjective, hence, $g = 0$ — a contradiction. Therefore, j_l^*A is simple. □

Lemma 1.3.2. *For every simple object $A_l \in \mathcal{C}_l$ there is a unique (up to an isomorphism) simple object $A \in \mathcal{C}(\Phi)$ such that $j_l^*A \simeq A_l$. Namely, $A = j_{l,!*}A_l$.*

Proof. The uniqueness is clear: if $A \in \mathcal{C}(\Phi)$ is simple and $j_l^*A \neq 0$, then the adjunction morphisms $j_{l,!}j_l^*A \to A$ and $A \to j_{l,*}j_l^*A$ are surjective and injective, respectively, and hence

$$A \simeq \operatorname{im}(\mu_l(j_l^*A) : j_{l,!}j_l^*A \to j_{l,*}j_l^*A) = j_{l,!*}(j_l^*A).$$

It remains to check that if $A_l \in \mathcal{C}_l$ is simple, then $j_{l,!*}(A_l)$ is simple. Let $B \subset j_{l,!*}(A_l)$ be a simple subobject. Since we have an inclusion $B \subset j_{l,*}A_l$ it follows from adjunction that j_l^*B is a non-zero subobject of A_l. Therefore, $j_l^*B = A_l$ and $B = j_{l,!*}(j_l^*B) = j_{l,!*}(A_l)$. □

1.4. **Gluing data for triangulated categories.** We refer to [3] for definitions concerning t-structures on triangulated categories.

Definition 1.4.1. Let $(\mathcal{D}_i)$ be a collection of triangulated categories with t-structures. A *gluing data* for $(\mathcal{D}_i)$ is a collection of exact functors

$$\mathcal{D}\Phi_{i,j} : \mathcal{D}_j \to \mathcal{D}_i \tag{1.4.1}$$

for all pairs (i,j) that are t-exact from the right (with respect to the given t-structures), such that $\mathcal{D}\Phi_{i,i} = \mathrm{Id}_{\mathcal{D}_i}$, and a collection of morphisms of functors

$$\mathcal{D}\nu_{i,j,k} : \mathcal{D}\Phi_{i,j} \circ \mathcal{D}\Phi_{j,k} \to \mathcal{D}\Phi_{i,k} \tag{1.4.2}$$

for all triples (i,j,k) such that $\mathcal{D}\nu_{i,i,k} = \mathrm{id}$, $\mathcal{D}\nu_{i,j,j} = \mathrm{id}$ and the analogue of the associativity equation (1.1.3) holds.

Let $\mathcal{C}_i$ be the heart of the t-structure on $\mathcal{D}_i$. It is easy to see that the functors

$$H^0\mathcal{D}\Phi_{i,j}|_{\mathcal{C}_i} = \tau_{\geq 0}\mathcal{D}\Phi_{i,j}|_{\mathcal{C}_j}$$

(where $\tau_{\geq 0}$ is the truncation with respect to the t-structure) extend to a gluing data for $\mathcal{C}_i$. Indeed, since $\mathcal{D}\Phi_{i,j}$ commutes with $\tau_{\geq 0}$ we have natural morphisms

$$\tau_{\geq 0} \circ \mathcal{D}\Phi_{i,j} \circ \tau_{\geq 0} \circ \mathcal{D}_{\Phi_{j,k}}|_{\mathcal{C}_k} \simeq \tau_{\geq 0} \circ \mathcal{D}\Phi_{i,j} \circ \mathcal{D}_{\Phi_{j,k}}|_{\mathcal{C}_k} \xrightarrow{\tau_{\geq 0}\mathcal{D}\nu_{i,j,k}} \tau_{\geq 0} \circ \mathcal{D}\Phi_{i,k}|_{\mathcal{C}_k}.$$

2. Grothendieck groups

2.1. **Formulation of the theorem.** Let $\mathcal{D}\Phi = (\mathcal{D}\Phi_{i,j}, \mathcal{D}\nu_{i,j,k})$ be the gluing data for the derived categories $(\mathcal{D}^b(\mathcal{C}_i))$ of the abelian categories $(\mathcal{C}_i)_{1\le i\le n}$, where $\mathcal{D}^b(\mathcal{C}_i)$ are equipped with standard t-structures. Let $(\Phi_{i,j} = H^0\mathcal{D}\Phi_{i,j}|_{\mathcal{C}_i}, \nu_{i,j,k})$ be the induced gluing data for $(\mathcal{C}_i)$. Then for every (i,j) there is an induced homomorphism of Grothendieck groups

$$\phi_{ij} = K_0(\Phi_{i,j}) : K_0(\mathcal{C}_j) \simeq K_0(\mathcal{D}_j) \to K_0(\mathcal{D}_i) \simeq K_0(\mathcal{C}_i).$$

Let us denote by $\mathcal{C}_{i,j}$ the full subcategory in $\mathcal{C}_i$ consisting of objects A such that the morphism $\nu_{i,j,i,A} : \Phi_{i,j}\Phi_{j,i}(A) \to A$ is zero.

Lemma 2.1.1. *$\mathcal{C}_{i,j}$ is an abelian subcategory in $\mathcal{C}_i$ closed under passing to quotients and subobjects.*

Proof. It is clear that $\mathcal{C}_{i,j}$ is closed under passing to subobjects. The statement about quotients follows from the fact that the functor $\Phi_{i,j}\Phi_{j,i}$ is right-exact. $\square$

Let us denote by $K_{i,j} \subset K_0(\mathcal{C}_i)$ the image of the natural homomorphism $K_0(\mathcal{C}_{i,j}) \to K_0(\mathcal{C}_i)$.

Theorem 2.1.2. *Let $\mathcal{D}\Phi$ be a gluing data for $(\mathcal{D}^b(\mathcal{C}_i))$, and Φ the corresponding gluing data for $(\mathcal{C}_i)$. Assume that all categories $\mathcal{C}_i$ are artinian and noetherian. Then the image of the natural map $K_0(\mathcal{C}(\Phi)) \to \oplus_{i=1}^n K_0(\mathcal{C}_i) : [(A_i, \alpha_{ij})] \mapsto ([A_i])$ coincides with the subgroup*

$$K(\Phi) := \{(c_i) \in \oplus_i K_0(\mathcal{C}_i) \mid \phi_{i,j}c_j - c_i \in K_{i,j}\}.$$

We need several lemmas for the proof.

2.2. **Homological lemmas.**

Lemma 2.2.1. *Assume that the category $\mathcal{C}_i$ is artinian and noetherian. Then for every collection of indices $j_1,\dots, j_k$ the image of the natural map*

$$K_0(\mathcal{C}_{i,j_1} \cap \ldots \cap \mathcal{C}_{i,j_k}) \to K_0(\mathcal{C}_i) \tag{2.2.1}$$

coincides with $K_{i,j_1} \cap \ldots \cap K_{i,j_k}$.

Proof. Since $\mathcal{C}_i$ is artinian and noetherian, by the Jordan—Hölder theorem $K_0(\mathcal{C}_i)$ is a free abelian group with the natural basis corresponding to isomorphism classes of simple objects in $\mathcal{C}_i$. Now the categories $\mathcal{C}_{i,j}$ are closed under passing to sub- and quotient-objects by Lemma 2.1.1. Hence $K_{i,j}$ is spanned by the classes of simple objects that belong to $\mathcal{C}_{i,j}$, while the image of the homomorphism (2.2.1) is spanned by the classes of simple objects lying in $\mathcal{C}_{i,j_1} \cap \ldots \cap \mathcal{C}_{i,j_k}$ and the assertion follows. $\square$

Lemma 2.2.2. *Let $\mathcal{D}$ be a triangulated category with a t-structure such that $\mathcal{D} = \cup_n \mathcal{D}^{\le n}$, $F : \mathcal{D} \to \mathcal{D}$ is an exact functor which is t-exact from the right, i.e., $F(\mathcal{D}^{\le 0}) \subset \mathcal{D}^{\le 0}$. Let $\nu : F \to \mathrm{Id}$ be a morphism of exact functors. Then for any object $X \in \mathcal{D}$ such that the morphism $\nu_X : FX \to X$ is zero the natural morphism $\tau_{\ge 0}F(H^n(X)) \to H^n(X)$ is zero for any n, where $H^n(X) = \tau_{\le n}\tau_{\ge n}(X)[n]$, $\tau_\cdot$ are the truncation functors associated with the t-structure.*

Proof. We may assume that $X \in \mathcal{D}^{\le k}$ for some k. Consider the following morphism of exact triangles:

$$\begin{tikzcd}
F(\tau_{\le k-1}X) \arrow[r] \arrow[d, "\nu"] & FX \arrow[r] \arrow[d, "0"] & F(\tau_{\ge k}X) \arrow[r] \arrow[d, "\nu"] & \ldots \\
\tau_{\le k-1}X \arrow[r] & X \arrow[r] & \tau_{\ge k}X \arrow[r] & \ldots
\end{tikzcd} \tag{2.2.2}$$

Since F is t-exact from the right we have $F(\tau_{\leq k-1}X) \in \mathcal{D}^{\leq k-1}$, hence $\mathrm{Hom}^{-1}(F(\tau_{\leq k-1}X), \tau_{\geq k}X) = 0$. This implies that all vertical arrows in the above diagram are zero, hence the conclusion holds for $n = k$ and the assumption holds for $\tau_{\leq k-1}X$, so we may proceed by induction. ☐

Lemma 2.2.3. *For any object A_j of $\mathcal{C}_j$ we have $[H^n\mathcal{D}\Phi_{i,j}A_j] \in K_{i,j}$ for $n \leq -1$. In paricular, $[\Phi_{i,j}A_j] - \phi_{i,j}[A_j] \in K_{i,j}$.*

Proof. Let us denote $A_i = \Phi_{i,j}A_j \in \mathcal{C}_i$, so that we have the morphism $\beta : \mathcal{D}\Phi_{j,i}A_i \to A_j$ induced by $\mathcal{D}\nu_{j,i,j}$. Consider the following morphism of exact triangles:

$$\begin{tikzcd}
\mathcal{D}\Phi_{i,j}\mathcal{D}\Phi_{j,i}(\tau_{\leq -1}\mathcal{D}\Phi_{i,j}A_j) \arrow[r] \arrow[d, "\gamma'"] & \mathcal{D}\Phi_{i,j}\mathcal{D}\Phi_{j,i}(\mathcal{D}\Phi_{i,j}A_j) \arrow[r, "\pi"] \arrow[d, "\gamma"] & \mathcal{D}\Phi_{i,j}\mathcal{D}\Phi_{j,i}(A_i) \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq -1}\mathcal{D}\Phi_{i,j}A_j \arrow[r] & \mathcal{D}\Phi_{i,j}A_j \arrow[r] & A_i \arrow[r] & \ldots
\end{tikzcd} \tag{2.2.3}$$

where the vertical arrows are induced by $\mathcal{D}\nu_{i,j,i}$. The associativity equation for $\mathcal{D}\nu$ implies that $\gamma = \delta \circ \pi$ where δ is the morphism

$$\delta = \mathcal{D}\Phi_{i,j}\beta : \mathcal{D}\Phi_{i,j}\mathcal{D}\Phi_{j,i}A_i \to \mathcal{D}\Phi_{i,j}A_j.$$

It follows that the composition

$$\mathcal{D}\Phi_{i,j}\mathcal{D}\Phi_{j,i}\tau_{\leq -1}\mathcal{D}\Phi_{i,j}A_j \xrightarrow{\gamma'} \tau_{\leq -1}\mathcal{D}\Phi_{i,j}A_j \to \mathcal{D}\Phi_{i,j}A_j$$

is zero. Since $\mathcal{D}\Phi_{i,j}\mathcal{D}\Phi_{j,i}\tau_{\leq -1}\mathcal{D}\Phi_{i,j}A_j \in \mathcal{D}^{\leq -1}$ we get that $\gamma' = 0$. By the previous Lemma this implies that $[H^n\mathcal{D}\Phi_{i,j}A_j] \in K_{i,j}$ for $n \leq -1$ as required. ☐

Lemma 2.2.4. *Let $\mathcal{C}^n(\Phi)$ be the full subcategory of $\mathcal{C}(\Phi)$ consisting of objects $(A_i; \alpha_{ij})$ with $A_n = 0$. Then $\mathcal{C}^n(\Phi)$ is equivalent to $\mathcal{C}(\Phi')$ for some new gluing data Φ' for $n-1$ categories $\mathcal{C}_{1,n}$,..., $\mathcal{C}_{n-1,n}$.*

Proof. If $(A_i; \alpha_{ij})$ is an object of $\mathcal{C}^n(\Phi)$, then the condition (1.1.4) implies that the composition

$$\Phi_{i,n}\Phi_{n,j}A_j \xrightarrow{\nu_{i,n,j}} \Phi_{i,j}A_j \xrightarrow{\alpha_{ij}} A_i$$

is zero for every pair (i, j), in particular, $A_i \in \mathcal{C}_{i,n}$ for every i. For every pair (i, j) such that $i, j \leq n-1$ and $i \neq j$, let us denote by $\Phi'_{i,j}$ the cokernel of the morphism of functors $\nu_{i,n,j} : \Phi_{i,n}\Phi_{n,j} \to \Phi_{i,j}$. Then we have a canonical morphism of functors $\Phi_{i,j} \to \Phi'_{i,j}$ and the above condition means that α_{ij} factors through a morphism $\alpha'_{ij} : \Phi'_{i,j}A_j \to A_i$. It is easy to see that the functor $\Phi'_{i,j}$ is right-exact (as the cokernel of right-exact functors). Let us check that $\Phi'_{i,j}$ sends $\mathcal{C}_j$ to $\mathcal{C}_{i,n}$. Indeed, for any $A_j \in \mathcal{C}_j$ the morphism

$$\nu_{i,n,i} : \Phi_{i,n}\Phi_{n,i}\Phi_{i,j}A_j \to \Phi_{i,j}A_j \tag{2.2.4}$$

factors through a morphism $\Phi_{i,n}\Phi_{n,i}\Phi_{i,j}A_j \to \Phi_{i,n}\Phi_{n,j}A_j$. Hence, the composition of (2.2.4) with the projection $\Phi_{i,j}A_j \to \Phi'_{i,j}A_j$ is zero which implies that $\Phi'_{i,j}A_j \in \mathcal{C}_{i,n}$. Furthermore, there are unique morphisms of functors $\Phi'_{i,j}\Phi'_{j,k} \to \Phi'_{i,k}$ compatible with $\nu_{i,j,k}$, so that we get a gluing data Φ' for $\mathcal{C}_{1,n}$,...,$\mathcal{C}_{n-1,n}$ such that $(A_i, i = 1, \ldots, n-1; \alpha'_{ij})$ is an object of $\mathcal{C}(\Phi')$. Clearly, this gives an equivalence of the category $\mathcal{C}^n(\Phi)$ with $\mathcal{C}(\Phi')$. ☐

2.3. **Proof of theorem 2.1.2.** Let us check first that the image of $K_0(\mathcal{C}(\Phi))$ is contained in $K(\Phi)$. It suffices to check that for every i and every $A \in \mathcal{C}(\Phi)$ one has $\phi_{i,n}[A_n] - [A_i] \in K_{i,n}$. We note that for every $A \in \mathcal{C}(\Phi)$ the kernel and the cokernel of the natural morphism $j_{n,!}j_n^*A \to A$ belong to $\mathcal{C}^n(\Phi)$. Hence, $K_0(\mathcal{C}(\Phi))$ is generated by the image of the map $K_0(\mathcal{C}^n(\Phi)) \to K_0(\mathcal{C}(\Phi))$ and by the classes of $j_{n,!}(A_n)$, $A_n \in \mathcal{C}_n$. Let us check the above condition for these two classes of elements separately. If $(A_i; \alpha_{ij}) \in$

$\mathcal{C}^n(\Phi)$, then by definition $A_i \in \mathcal{C}_{i,n}$ for any i, while $A_n = 0$; hence $\phi_{i,n}[A_n] - [A_i] = -[A_i] \in K_{i,n}$ for all i. On the other hand, if $(A_i; \alpha_{ij}) = j_{n,!}(A_n)$, then $A_i = \Phi_{i,n} A_n$ and

$$\phi_{i,n}[A_n] - [A_i] = \phi_{i,n}[A_n] - [\Phi_{i,n} A_n] \in K_{i,n}$$

by Lemma 2.2.3. Thus, the condition $\phi_{i,n}[A_n] - [A_i] \in K_{i,n}$ is satisfied for all objects of $\mathcal{C}(\Phi)$.

It remains to check that the map $K_0(\mathcal{C}(\Phi)) \to K(\Phi)$ is surjective. Let us denote by $\mathcal{C}^l(\Phi)$ the full subcategory of $\mathcal{C}(\Phi)$ consisting of all objects $(A_i; \alpha_{ij})$ with $A_i = 0$ for $i \geq l$. Similarly, let $K^l(\Phi)$ be the subgroup of elements (c_i) in $K(\Phi)$ with $c_i = 0$ for $i \geq l$. Note that the image of $K_0(\mathcal{C}^l(\Phi))$ is contained in $K^l(\Phi)$. Let p_j be the projection of $\oplus_i K_0(\mathcal{C}_i)$ on its j-th factor. Then $p_{l-1}(K^l(\Phi)) \subset \cap_{j \geq l} K_{l-1,j}$ since for $(c_i) \in K^l(\Phi)$ and $j \geq l$ we have $c_{l-1} = c_{l-1} - \phi_{l-1,j} c_j \in K_{l-1,j}$. Thus, we have an exact sequence

$$0 \to K^{l-1}(\Phi) \to K^l(\Phi) \to \cap_{j \geq l} K_{l-1,j}.$$

Hence, to prove the surjectivity of the map $K_0(\mathcal{C}(\Phi)) \to K(\Phi)$ it is sufficient to check that the map

$$K_0(\mathcal{C}^l(\Phi)) \to \cap_{j \geq l} K_{l-1,j} : [(A_i; \alpha_{ij})] \mapsto [A_{l-1}] \tag{2.3.1}$$

is surjective for each l. Note that for every gluing data Φ (not necessarily extending to the derived categories) the natural homomorphism $K_0(\mathcal{C}(\Phi)) \to K_0(\mathcal{C}_n)$ is surjective ($[j_{n,!}(A_n)]$ maps to $[A_n]$). Now the iterated application of Lemma 2.2.4 gives an equivalence of $\mathcal{C}^l(\Phi)$ with $\mathcal{C}(\Phi')$ for some gluing data Φ' on $l-1$ categories $\cap_{j \geq l} \mathcal{C}_{1,j}$,..., $\cap_{j \geq l} \mathcal{C}_{l-1,j}$, such that the map (2.3.1) is identified with the homomorphism

$$K_0(\mathcal{C}(\Phi')) \to K_0(\cap_{j \geq l} \mathcal{C}_{l-1,j}) \to \cap_{j \geq l} K_{l-1,j}$$

which is surjective by Lemma 2.2.1. □

2.4. **Gluing of finite type.**

Proposition 2.4.1. *Assume that every functor $\Phi_{i,j}$ has the right adjoint $\Psi_{j,i}$ and that the categories $\mathcal{C}_i$ are artinian and noetherian. Then the natural homomorphism $K_0(\mathcal{C}(\Phi)) \to \oplus_i K_0(\mathcal{C}_i)$ is injective.*

Proof. Since the category $\mathcal{C}(\Phi)$ is artinian and noetherian, it follows that $K_0(\mathcal{C}(\Phi))$ is the free abelian group with the basis $[A]$, where A runs through the isomorphism classes of simple objects in $\mathcal{C}(\Phi)$. Now the assertion follows immediately from Lemmas 1.3.1 and 1.3.2. □

Definition 2.4.2. We say that Φ is a gluing data of *finite type* if all the categories $\mathcal{C}_i$ are artinian and noetherian, and every functor Φ_{ij} has the right adjoint Ψ_{ji}.

Combining Proposition 2.4.1 with Theorem 2.1.2 we obtain the following result.

Theorem 2.4.3. *Let $\mathcal{D}\Phi$ be a gluing data for $(\mathcal{D}^b(\mathcal{C}_i))$, and Φ the corresponding gluing data for $(\mathcal{C}_i)$. Assume that Φ is of finite type. Then the natural map $K_0(\mathcal{C}(\Phi)) \to \oplus_{i=1}^n K_0(\mathcal{C}_i) : [(A_i, \alpha_{ij})] \mapsto ([A_i])$ induces an isomorphism $K_0(\mathcal{C}(\Phi)) \simeq K(\Phi)$.*

3. Gluing for Coxeter groups

3.1. W**-gluing.**

Let (W, S) be a finite Coxeter group, and $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function. Let $(\mathcal{C}_w; w \in W)$ be a collection of abelian categories.

Definition 3.1.1. A W-*gluing data* is a gluing data $\Phi_w : \mathcal{C}_{w'} \to \mathcal{C}_{ww'}$, $\nu_{w,w'} : \Phi_w \circ \Phi_{w'} \to \Phi_{ww'}$ for $(\mathcal{C}_w)$ such that $\nu_{w,w'}$ is an isomorphism for every pair $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$.

Note that the condition of finiteness of W is imposed only because we usually consider the gluing of a finite number of categories. One can similarly treat the case of infinite number of categories and define W-gluing data in this context.

3.2. **Quasi-actions of Coxeter groups.**

Definition 3.2.1. A *quasi-action* of a monoid M on a category $\mathcal{C}$ is a collection of functors $T(f)$, $f \in M$, from $\mathcal{C}$ to itself and of morphisms of functors $c_{f,g} : T(f) \circ T(g) \to T(fg)$, $f, g \in M$ satisfying the associativity condition.

An *action* of a monoid is a quasi-action such that all morphisms $c_{f,g}$ are isomorphisms. Let (W, S) be a Coxeter system. The above definition of W-gluing data can be reformulated as follows: there is a quasi-action of W on $\oplus_{w\in W}\mathcal{C}_w$ given by functors $\Phi_w : \mathcal{C}_{w'} \to \mathcal{C}_{ww'}$ such that $\Phi_1 = \mathrm{Id}$ and $c_{w,w'}$ are isomorphisms when $\ell(ww') = \ell(w) + \ell(w')$.

Let us denote by B the generalized braid group corresponding to (W, S) and by $B^+ \subset B$ the positive braids submonoid (see [9]). Let us denote by $b \mapsto \overline{b}$ the natural homomorphism $B \to W$ and by $\tau : W \to B^+$ its canonical section such that τ is the identity on S and $\tau(ww') = \tau(w)\tau(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$. According to the main theorem of [9], to give an action of B^+ on a category $\mathcal{C}$, it is sufficient to have functors $T(w)$ corresponding to elements $\tau(w) \in B^+$ and isomorphisms $c_{w,w'} : T(w) \circ T(w') \widetilde{\to} T(ww')$ for pairs $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$, satisfying the associativity condition for triples $w, w', w'' \in W$, such that $\ell(ww'w'') = \ell(w) + \ell(w') + \ell(w'')$. It follows that a W-gluing data induces an action of B^+ on $\oplus_w\mathcal{C}_w$. Conversely, assume that we are given the functors $\mathcal{C}_w$ and isomorphisms $c_{w,w'}$ only for (w, w') with $\ell(ww') = \ell(w) + \ell(w')$ so that we have an action of B^+ on $\oplus_w\mathcal{C}_w$. To get a quasi-action of W we have to give in addition some morphisms $c_{s,s} : \Phi_s\Phi_s \to \mathrm{Id}$. The natural question arises as to what compatibility conditions relating $c_{s,s}$ and the action of B^+ one should impose to obtain a quasi-action of W. The answer is given by the following theorem.

Theorem 3.2.2. *Assume that we have an action of B^+ on a category $\mathcal{C}$ given by the collection of functors $T(w)$, $w \in W$ (such that $T(1) = \mathrm{Id}$), and isomorphisms $c_{w,w'} : T(w)T(w') \widetilde{\to} T(ww')$ for $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$. Let $c_{s,s} : T(s)T(s) \to \mathrm{Id}$, $s \in S$, be a collection of morphisms satisfying the following two conditions:*

1. *For every $s \in S$ the following associativity equation holds:*

$$T(s)c_{s,s,X} = c_{s,s,T(s)X} : T(s)^3X \to T(s)X. \tag{3.2.1}$$

2. *For every $w \in W$ and $s, s' \in S$ such that $sw = ws'$ and $\ell(sw) = \ell(w) + 1$ the following diagram is commutative:*

$$\begin{tikzcd} T(s)T(w)T(s') \arrow[r] \arrow[d] & T(w)T(s')^2 \arrow[d, "c_{s',s'}"] \\ T(s)^2T(w) \arrow[r, "c_{s,s}"] & T(w) \end{tikzcd} \tag{3.2.2}$$

where the unmarked arrows are induced by $c_{s,w}$ and $c_{w,s'}$.

Then there is a canonical quasi-action of W on $\mathcal{C}$.

Proof. Following [4] we denote by P_s the set of $w \in W$ such that $\ell(sw) = \ell(w) + 1$. Let us first construct the canonical morphisms $c_{s,w} : T(s)T(w) \to T(sw)$ for every $s \in S$, $w \in W$. When $w \in P_s$ they are given by the structure of B^+-action. Otherwise $sw \in P_s$ and we have the morphism $c_{s,w} : T(s)T(w) \widetilde{\to} T(s)T(s)T(sw) \to T(sw)$ induced by $c^{-1}_{s,sw}$ and $c_{s,s}$. It is easy to check using (3.2.1) that the

following triangle is commutative for every $s \in S$, $w \in W$:

$$
\begin{tikzcd}
T(s)^2T(w) \arrow[d, "c_{s,w}"] \arrow[dr, "c_{s,s}"] & \\
T(s)T(sw) \arrow[r, "c_{s,ws}"] & T_w
\end{tikzcd}
\tag{3.2.3}
$$

Similarly, one defines morphisms $c_{w,s} : T(w)T(s) \to T(ws)$ for every $s \in S$, $w \in W$.

We claim that these morphisms satisfy the following associativity condition: for any $w \in W$, $s, s' \in S$ the diagram

$$
\begin{tikzcd}
T(s)T(w)T(s') \arrow[r, "c_{w,s'}"] \arrow[d, "c_{s,w}"] & T(s)T(ws') \arrow[d, "c_{s,ws'}"] \\
T(sw)T(s') \arrow[r, "c_{sw,s'}"] & T(sws')
\end{tikzcd}
\tag{3.2.4}
$$

is commutative. Assume at first that $w \in P_s$. Consider two cases.

1. $ws' \in P_s$. If $\ell(ws') = \ell(w) + 1$ then $\ell(sws') = \ell(w) + 2$ and the required associativity holds by definition of the B^+-action. Otherwise, $\ell(w) = \ell(ws') + 1$, and hence $\ell(sws') = \ell(ws') + 1 = \ell(w) = \ell(sw) - 1$, $T(w) \simeq T(ws')T(s')$, $T(sw) \simeq T(sws')T(s')$, and we are reduced to the commutativity of the diagram

$$
\begin{tikzcd}
T(s)T(ws')T(s')^2 \arrow[r, "c_{s',s'}"] \arrow[d, "c_{s,ws'}"] & T(s)T(ws') \arrow[d, "c_{s,ws'}"] \\
T(sws')T(s')^2 \arrow[r, "c_{s',s'}"] & T(sws')
\end{tikzcd}
\tag{3.2.5}
$$

which is clear.

2. $w \in P_s$, $ws' \notin P_s$. In this case according to [4], IV, 1.7 we have $sw = ws'$, so the required associativity follows from (3.2.2).

Thus, the diagram (3.2.4) is commutative for $w \in P_s$. Now assume that $w = sw'$ with $w' \in P_s$. Consider the following diagram:

$$
\begin{tikzcd}
T(s)T(sw')T(s') \arrow[r, "c_{s,w'}^{-1}"] \arrow[dr, "c_{s,sw'}"] & T(s)^2T(w')T(s') \arrow[r, "c_{w',s'}"] \arrow[d, "c_{s,s}"] & T(s)^2T(w's') \arrow[r, "c_{s,w's'}"] \arrow[d, "c_{s,s}"] & T(s)T(sw's') \arrow[dl, "c_{s,sw's'}"] \\
& T(w')T(s') \arrow[r, "c_{w',s'}"] & T(w's') &
\end{tikzcd}
\tag{3.2.6}
$$

In this diagram the square is commutative, the left triangle is commutative by definition of cs, sw' and the right triangle is commutative by (3.2.3). Also the commutativity of (3.2.4) for w' implies that the composition of the top arrows coincides with the top arrow in the diagram (3.2.4) for w and hence, the commutativity of (3.2.4) for w.

Now for any $w \in W$ with a reduced decomposition $w = s_{i_1}s_{i_2}\dots s_{i_l}$ which we denote by $\mathbf{s}$ and any $w' \in W$, we define the morphism $c_{\mathbf{s},w'} : T(w)T(w') \to T(ww')$ inductively as the composition

$$
c_{\mathbf{s},w'} : T(w)T(w') \simeq T(s_{i_1})T(s_{i_2}\dots s_{i_l})T(w') \to T(s_{i_1})T(s_{i_2}\dots s_{i_l}w') \to T(ww')
$$

where the latter arrow is $c_{s_{i_1},s_{i_2}\dots s_{i_l}w'}$. We are going to prove that $c_{\mathbf{s},w'}$ does not depend on a choice of the reduced decomposition $\mathbf{s}$ of w. First we note that the following diagram is commutative for every reduced decomposition $\mathbf{s}$ of w and every $s \in S$, $w' \in W$:

$$\begin{tikzcd} T(w)T(w')T(s) \arrow[r, "c_{w,s}"] \arrow[d, "c_{\mathbf{s},w'}"] & T(w)T(w's) \arrow[d, "c_{\mathbf{s},w's}"] \\ T(ww')T(s) \arrow[r, "c_{ww',s}"] & T(ww's) \end{tikzcd} \tag{3.2.7}$$

Indeed, this follows from (3.2.4) by induction in the length of w. Now the required independence of $c_{\mathbf{s},w'}$ on a choice of $\mathbf{s}$ follows from (3.2.7) by induction in the length of w' (the base of the induction is the case $w' = 1$ when the assertion is obvious). Let us denote $c_{w,w'} = c_{\mathbf{s},w'}$ for any reduced decomposition $\mathbf{s}$ of w. Using (3.2.7) it is easy to show that one would obtain the same morphism starting with a reduced decomposition of w'. It remains to check that the following diagram is commutative for any $w, w_1, w_2 \in W$:

$$\begin{tikzcd} T(w_1)T(w)T(w_2) \arrow[r, "c_{w,w_2}"] \arrow[d, "c_{w_1,w}"] & T(w_1)T(ww_2) \arrow[d, "c_{w_1,ww_2}"] \\ T(w_1w)T(w_2) \arrow[r, "c_{w_1w,w_2}"] & T(w_1ww_2) \end{tikzcd} \tag{3.2.8}$$

When $\ell(w_2) = 1$ this reduces to (3.2.7). The general case follows easily by induction in $\ell(w_2)$. □

3.3. **Reduction to rank-2 subgroups.** The condition (2) of Theorem 3.2.2 can be reduced to rank-2 subgroups of W using the following result. For every pair (s_1, s_2) of elements of S, such that the order of s_1s_2 is $2m + \varepsilon$ with $\varepsilon \in \{0, 1\}$, let us denote $s = s(s_1, s_2) = s_1$, $s' = s'(s_1, s_2) = s_1$ if $\varepsilon = 0$, and $s' = s_2$ if $\varepsilon = 1$. Let also $w = w(s_1, s_2) = (s_2s_1)^{n-\varepsilon}s_2^{\varepsilon}$. Then $sw = ws'$ and $\ell(sw) = \ell(w) + 1$.

Proposition 3.3.1. *We keep the notation and assumptions of Theorem 3.2.2. Assume that the diagram (3.2.2) is commutative for all the triples $s(s_1, s_2), s'(s_1, s_2), w(s_1, s_2)$ associated with pairs (s_1, s_2) of elements of S as above. Then it is commutative for all (w, s, s') such that $sw = ws'$ and $\ell(sw) = \ell(w)+1$.*

Lemma 3.3.2. *Let (W, S) be a Coxeter system. Assume that $sw = ws'$, where $w \in W$, $s, s' \in S$, $\ell(sw) = \ell(w) + 1$, $w \neq 1$. Then there exists an element $s_2 \in S$, such that for the element $w(s_1, s_2)$ associated with the pair $(s_1 = s, s_2)$, one has $\ell(w) = \ell(w(s_1, s_2)) + \ell(w(s_1, s_2)^{-1}w)$.*

Proof. Consider a reduced decomposition $(s_2, \dots , s_n)$ of w. Then we have two reduced decompositions of $sw = ws'$: $\mathbf{s} = (s_1 = s, s_2, \dots , s_n)$ and $\mathbf{s}' = (s_2, \dots , s_n, s')$. Now the second sequence is obtained from the first one by a series of standard moves associated with braid relations for couples of elements of S. If the first move in such a series does not touch the first member of $\mathbf{s}$, then it just changes a reduced decomposition of w. Thus, we can choose an initial reduced decomposition for W in such a way that the first move does touch s_1. Then we have $w = w(s_1, s_2)w'$ where $\ell(w) = \ell(w(s_1, s_2)) + \ell(w')$ as required. □

Proof of Proposition 3.3.1. Induction on the length of w and Lemma 3.3.2 show that it is sufficient to prove the following: if $w = w_1w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$ and one has $s_1w_1 = w_1s_2$, $s_2w_2 = w_2s_3$ where $\ell(s_1w) = \ell(w) + 1$, then the commutativity of the diagram (3.2.2) for the triples (w_1, s_1, s_2) and

(w_2, s_2, s_3) implies its commutativity for (w, s_1, s_3). This can be easily checked using the fact that we have a B^+-action. $\square$

3.4. **Grothendieck groups.** As before we can define a notion of W-gluing data $\mathcal{D}\Phi$ for the derived categories $(\mathcal{D}^b(\mathcal{C}_w))$, which induce the W-gluing data Φ for $(\mathcal{C}_w)$. Let us denote by $\phi_w : K_0(\mathcal{C}'_w) \to K_0(\mathcal{C}_{ww'})$ the corresponding homomorphisms of Grothendieck groups. If in addition the categories $\mathcal{C}_w$ are artinian and noetherian, then according to Theorem 2.1.2 the image of the homomorphism $K_0(\mathcal{C}(\Phi)) \to \oplus_w K_0(\mathcal{C}_w)$ coincides with the subgroup

$$K(\Phi) = \{(c_w) \in \oplus_{w \in W} K_0(\mathcal{C}_w) \mid \phi_w c_{w'} - c_{ww'} \in K_{ww',w'},\ w, w' \in W\}.$$

The property that $\nu_{w,w'}$ is an isomorphism when $\ell(ww') = \ell(w) + \ell(w')$ allows us to give an alternative definition for the category $\mathcal{C}(\Phi)$. Namely, the morphisms $\alpha_{w',w} : \Phi_{w'} A_w \to A_{w'w}$ for all $w' \in W$ can be recovered uniquely from the morphisms $\alpha_{s,w} : \Phi_s A_w \to A_{sw}$ provided the latter morphisms respect the relations in W in the obvious sense. On the level of Grothendieck groups this is reflected in the following result.

Proposition 3.4.1. *Let $\mathcal{D}\Phi$ be a W-gluing data for $(\mathcal{D}^b(\mathcal{C}_w))$. Then*

$$K(\Phi) = \{(c_w) \in \oplus_{w \in W} K_0(\mathcal{C}_w) \mid \phi_s c_w - c_{sw} \in K_{sw,w},\ w \in W, s \in S\}. \tag{3.4.1}$$

Proof. It is clear that the left-hand side of (3.4.1) is contained in the right-hand side, so we have to check the inverse inclusion. If $\ell(sw) = \ell(w) + 1$, then $\phi_{sw} = \phi_s \phi_w$. Hence,

$$\phi_{sw} c_{w'} - c_{sww'} = (\phi_s c_{ww'} - c_{sww'}) + \phi_s(\phi_w c_{w'} - c_{ww'}).$$

Thus, it is sufficient to prove that

$$K_{sww',ww'} + \phi_s(K_{ww',w'}) \subset K_{sww',w'}$$

provided that $\ell(sw) = \ell(w) + 1$. By definition $\mathcal{C}_{sww',w'}$ consists of objects $X \in \mathcal{C}_{sww'}$ for which the morphism

$$\Phi_s \Phi_w \Phi_{w^{-1}} \Phi_s X \to X \tag{3.4.2}$$

is zero. Since this morphism factors through $\Phi_s \Phi_s X \to X$ we have an obvious inclusion $\mathcal{C}_{sww',ww'} \subset \mathcal{C}_{sww',w'}$. By Lemma 2.2.3 for $Y \in \mathcal{C}_{ww'}$ we have $[H^n \mathcal{D}\Phi_s Y] \in K_{sww',ww'}$ for $n \leq -1$. Thus, it is sufficient to prove the inclusion $\Phi_s(\mathcal{C}_{ww',w'}) \subset \mathcal{C}_{sww',w'}$. But for $X = \Phi_s Y$ the morphism (3.4.2) factorizes as follows:

$$\Phi_s \Phi_w \Phi_{w^{-1}} \Phi_s \Phi_s Y \to \Phi_s \Phi_w \Phi_{w^{-1}} Y \to \Phi_s Y.$$

If $Y \in \mathcal{C}_{ww',w'}$, then the latter arrow is zero, hence, $\Phi_s Y \in \mathcal{C}_{sww',w'}$. $\square$

4. Symplectic Fourier transform

4.1. **Functors and distinguished triangles.** Let k be a field of characteristic $p > 0$ that is either finite or algebraically closed, and S a scheme of finite type over k. Let $\pi : V \to S$ be a symplectic vector bundle of rank $2n$ over S, and $\langle , \rangle : V \times_S V \to \mathbb{G}_a$ the corresponding symplectic pairing. Let us fix a non-trivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_l^*$. The Fourier—Deligne transform $\mathcal{F} = \mathcal{F}_\psi$ is the involution of $\mathcal{D}_c^b(V, \overline{\mathbb{Q}}_l)$ defined by

$$\mathcal{F}(K) = p_{2!}(\mathcal{L} \otimes p_1^*(K))[2n](n).$$

where p_i are the projections of the product $V \times_S V$ on its factors, and $\mathcal{L} = \mathcal{L}_\psi(\langle , \rangle)$ is a smooth rank-1 $\overline{\mathbb{Q}}_l$-sheaf on $V \times_S V$ which is the pullback of the Artin—Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{G}_a$ under the morphism $\langle , \rangle$. Let $s : S \to V$ be the zero section, $j : U \to V$ the complementary open subset to $s(S)$, and $p = \pi \circ j : U \to S$ the projection of U to S. Let us denote

$$\mathcal{F}_! = j^* \mathcal{F} j_! : \mathcal{D}_c^b(U, \overline{\mathbb{Q}}_l) \to \mathcal{D}_c^b(U, \overline{\mathbb{Q}}_l).$$

Proposition 4.1.1. *For every $K \in \mathcal{D}^b_c(U, \overline{\mathbb{Q}}_l)$ there is a canonical distinguished triangle in $\mathcal{D}^b_c(U, \overline{\mathbb{Q}}_l)$:*

$$\mathcal{F}^2_!(K) \to K \to p^* p_! K[4n](2n) \to \dots \tag{4.1.1}$$

Proof. Note that $\mathcal{F}_!(K) \simeq p_{2!}(\mathcal{L}' \otimes p_1^* K)[2n](n)$ where $\mathcal{L}' = \mathcal{L}|_{U^2}$; hence,

$$\mathcal{F}^2_!(K) \simeq p_{2!}((\mathcal{L}' \circ \mathcal{L}') \otimes p_1^* K)[4n](2n)$$

where $\mathcal{L}' \circ \mathcal{L}' = p_{13!}(p_{12}^* \mathcal{L}' \otimes p_{23}^* \mathcal{L}') \in \mathcal{D}^b_c(U^2, \overline{\mathbb{Q}}_l)$. Let $k = \mathrm{id} \times j \times \mathrm{id} : V \times_S U \times_S V \to V^3$ be the open embedding. Then

$$\mathcal{L}' \circ \mathcal{L}' \simeq (j \times j)^* p_{13!} k_! k^* (p_{12}^* \mathcal{L} \otimes p_{23}^* \mathcal{L}).$$

Note that we have a canonical isomorphism

$$\mathcal{L}_\psi(\langle x_1, x_2 \rangle) \otimes \mathcal{L}_\psi(\langle x_2, x_3 \rangle) \simeq \mathcal{L}_\psi(\langle x_2, x_3 - x_1 \rangle)$$

on V^3, and the latter rank-1 local system is trivial on the complement to k; hence we have the following exact triangle:

$$\mathcal{L}' \circ \mathcal{L}' \to (j \times j)^* (\mathcal{L} \circ \mathcal{L}) \to \overline{\mathbb{Q}}_{l, U^2} \to \dots ,$$

where $\mathcal{L} \circ \mathcal{L} = p_{13!}(p_{12}^* \mathcal{L} \otimes p_{23}^* \mathcal{L}) \simeq \Delta_* \overline{\mathbb{Q}}_{l,V}[-4n](-2n)$, which induces the exact triangle (4.1.1). □

Let us denote by ${}^p\mathcal{F}_! = {}^pH^0\, \mathcal{F}_! : \mathrm{Perv}(U) \to \mathrm{Perv}(U)$ the right-exact functor induced by $\mathcal{F}_!$. Then we have a canonical morphism of functors $\nu : ({}^p\mathcal{F}_!)^2 \to \mathrm{Id}$.

Corollary 4.1.2. *The full subcategory of objects $K \in \mathrm{Perv}(U)$ such that $\nu_K : ({}^p\mathcal{F}_!)^2 K \to K$ is zero coincides with the subcategory $p^*[2n](\mathrm{Perv}(S)) \subset \mathrm{Perv}(U)$.*

Proof. For $K \in \mathrm{Perv}(U)$ the exact triangle (4.1.1) induces the long exact sequence

$$\dots \to ({}^pF_!)^2 K \to K \to p^*[2n]({}^pH^{2n}\, p_! K)(2n) \to 0.$$

Since ${}^pH^{4n}\, p_! p^*(2n) \simeq \mathrm{Id}_{\mathrm{Perv}(S)}$, the assertion follows. □

4.2. **Associativity.** To check the associativity condition (3.2.1) for the above morphism $\mathcal{F}^2_! \to \mathrm{Id}$, it is sufficient to check this condition for the morphism $\mathcal{F}^2 \to \mathrm{Id}$, which is done in the following proposition.

Proposition 4.2.1. *Let $\mathcal{F}$ be the Fourier transform associated with a symplectic vector bundle V of rank $2n$. Then the canonical morphism of functors $c : \mathcal{F}^2 \to \mathrm{Id}$ satisfies the associativity condition $\mathcal{F} \circ c = c \circ \mathcal{F} : \mathcal{F}^3 \to \mathcal{F}$.*

Proof. Recall that $\mathcal{F}$ is given by the kernel $\mathcal{L}[2n](n)$ on V^2 and c is induced by the canonical morphisms of kernels

$$p_{13!}(p_{12}^* \mathcal{L} \otimes p_{23}^* \mathcal{L}[4n](2n)) \to p_{13!}(\Delta_{13*} \Delta_{13}^* (p_{12}^* \mathcal{L} \otimes p_{23}^* \mathcal{L}[4n](2n))) \simeq$$
$$\simeq p_{13!} \Delta_{13*} \overline{\mathbb{Q}}_{l, V^2}[4n](2n) \xrightarrow{\mathrm{Tr}} \Delta_* \overline{\mathbb{Q}}_{l,V},$$

where p_{ij} are the projections of V^3 on the double products, $\Delta : V \to V^2$ and $\Delta_{13} : V^2 \to V^3$ are the diagonals, Tr is induced by the trace morphism (see [14]) (the triviality of $\Delta_{13}^*(p_{12}^* \mathcal{L} \otimes p_{23}^* \mathcal{L})$ follows from the skew-symmetry of $\mathcal{L}$). Now $\mathcal{F}^3$ is given by the kernel $p_{14!}(L[6n](3n))$ on V^2 where $L = p_{12}^* \mathcal{L} \otimes p_{23}^* \mathcal{L} \otimes p_{34}^* \mathcal{L}$ p_{ij} are the projections of V^4 on the double products, and the associativity boils down to the

commutativity of the diagram

$$\begin{tikzcd}[column sep=small]
& p_{14!}(L[6n](3n)) \arrow[dl] \arrow[dr] & \\
p_{14!}(\Delta_{13*}\Delta_{13}^{*}L[6n](3n)) \arrow[dr, "\mathrm{Tr}"] & & p_{14!}(\Delta_{24*}\Delta_{24}^{*}L[6n](3n)) \arrow[dl, "\mathrm{Tr}"] \\
& \mathcal{L}[2n](n) &
\end{tikzcd} \tag{4.2.1}$$

where $\Delta_{ij} : V^3 \to V^4$ are the diagonals, and the lower diagonal arrows are induced by the isomorphisms $\Delta_{13}^{*}L \simeq \Delta_{24}^{*}L \simeq p_{13}^{*}\mathcal{L}$ on V^3 and by the trace morphisms. Changing the coordinates on V^4 by $(x, y, z, t) \mapsto (x, y - t, z - x, t)$ one can see that this reduces to the commutativity of the diagram

$$\begin{tikzcd}
p_!(\mathcal{L}[4n](2n)) \arrow[r] \arrow[d] & p_!(i_{1*}i_1^{*}\mathcal{L}[4n](2n)) \simeq \quad p_!(\overline{\mathbb{Q}}_{l,V}[4n](2n)) \arrow[d, "\mathrm{Tr}"] \\
p_!(i_{2*}i_2^{*}\mathcal{L}[4n](2n)) \quad \simeq p_!(\overline{\mathbb{Q}}_{l,V}[4n](2n)) \arrow[r, "\mathrm{Tr}"] & \overline{\mathbb{Q}}_l
\end{tikzcd} \tag{4.2.2}$$

where $i_1 : V \times 0 \to V^2$, $i_2 : 0 \times V \to V^2$ are the natural closed embeddings, and p denotes the projection to $\mathrm{Spec}(k)$. Here is the argument due to M. Rapoport verifying the commutativity of (4.2.2). Obviously, we may assume that the base is a point and $n = 1$, so that $V = \mathbb{A}^2$. Let $i : I \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^2$ be the tautological line bundle over the projective line (the incidence correspondence), and $p_2^{*}\mathcal{L}$ be the pullback of $\mathcal{L}$ under the projection $p_2 : \mathbb{P}^1 \times \mathbb{A}^2 \to \mathbb{A}^2$. Then i is an embedding of a lagrangian subbundle in the trivial rank-2 symplectic bundle over $\mathbb{P}^1$; hence, we have a sequence of canonical morphisms

$$q^{*}p_!(\mathcal{L}[4](2)) \widetilde{\to} \pi_!(p_2^{*}\mathcal{L}[4](2)) \to \pi_!(i_{*}i^{*}p_2^{*}\mathcal{L}[4](2)) \widetilde{\to} \pi_!(\overline{\mathbb{Q}}_{l,I}[4](2)) \widetilde{\to} q^{*}\overline{\mathbb{Q}}_l$$

where π denotes the projection to $\mathbb{P}^1$, and q is the projection of $\mathbb{P}^1$ to $\mathrm{Spec}(k)$. Let $\phi : q^{*}p_!(\mathcal{L}[4](2)) \to q^{*}\overline{\mathbb{Q}}_l$ be the composed morphism. Then $\phi = q^{*}\widetilde{\phi}$ for some morphism $\widetilde{\phi} : p_!(\mathcal{L}[4](2)) \to \overline{\mathbb{Q}}_l$ (since in fact $p_!(\mathcal{L}[4](2)) \simeq \overline{\mathbb{Q}}_l$). On the other hand, the two morphisms $p_!(\mathcal{L}[4](2)) \to \overline{\mathbb{Q}}_l$ in (4.2.2) are the restrictions of ϕ to the points $x, y \in \mathbb{P}^1$ corresponding to the coordinate lines in $\mathbb{A}^2$; hence, they are both equal to $\widetilde{\phi}$. □

5. Gluing on the basic affine space

5.1. **Setup.** Let k be a field of characteristic $p > 2$, which is either finite or algebraically closed, let G be a connected, simply-connected, semisimple algebraic group over k. Assume that G is split over k and fix a split maximal torus $T \subset G$ and a Borel subgroup B containing T. Also we denote by $W = N(T)/T$ the Weil group of G, and by $S \subset W$ the set of simple reflections. Let $X = G/U$ be the corresponding basic affine space, where U is the unipotent radical of B. Following [11] we are going to construct a W-gluing data such that $\mathcal{C}_w$ is the category $\mathrm{Perv}(X)$ of perverse sheaves on X for every w. To define the gluing functors we need some additional data. First, we fix a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}_l}^{*}$. We denote by $\mathcal{L}_\psi$ the corresponding Artin—Schreier sheaf on $\mathbb{G}_{a,\mathbb{F}_p}$. Second, for every simple root α_s we fix an isomorphism of the corresponding 1-parameter subgroup $U_s \subset U$ with the additive group $\mathbb{G}_{a,k}$. This

defines uniquely a homomorphism $\rho_s : \mathrm{SL}_{2,k} \to G$ such that φ_s induces the given isomorphism of $\mathbb{G}_{a,k}$ (embedded in $\mathrm{SL}_{2,k}$ as upper-triangular matrices) with U_s (see [11]). Let

$$n_s = \rho_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For every $w \in W$ with a reduced decomposition $w = s_1 \dots s_l$ we set $n_w = n_{s_1} \dots n_{s_l} \in G$. Then n_w does not depend on a choice of a reduced decomposition, so we get a canonical system of representatives for W in $N(T)$.

5.2. **Gluing functors.** For every element $w \in W$ we consider the subtorus

$$T_w = \prod_{\alpha \in R(w)} \alpha^\vee(\mathbb{G}_m) \subset T$$

where $R(w) \subset R^+$ is the set of positive roots $\alpha \in R^+$ such that $w(\alpha) \in -R^+$, $\alpha^\vee$ is the corresponding coroot. By definition we have a surjective homomorphism

$$\prod_{\alpha \in R(w)} \alpha^\vee : \mathbb{G}_{m,k}^{R(w)} \to T_w.$$

It is easy to see that

$$T_w = \prod_{s \in S_w} T_s$$

where $S_w \subset S$ is the set of simple reflections s such that $s \leq w$ with respect to the Bruhat order (see [11], 2.2.1). Now we define $X(w) \subset X \times_k X$ as the subvariety of pairs $(gU, g'U) \subset X \times_k X$ such that $g^{-1}g' \in U n_w T_w T$. There is a canonical projection $\mathrm{pr}_w : X(w) \to T_w$ sending $(gU, g'U)$ to the unique $t_w \in T_w$ such that $g^{-1}g' \in U n_w t_w U$. The morphism pr_w is smooth of relative dimension $\dim X + \ell(w)$, surjective, with connected geometric fibers. The last ingredient in the definition of the gluing functors is the morphism

$$\sigma_w : \mathbb{G}_{m,k}^{R(w)} \to \mathbb{G}_{a,k} : (z_\alpha)_{\alpha \in R(w)} \mapsto -\sum_{\alpha \in R(w)} z_\alpha.$$

Now we set

$$K(w) = K_\psi(w) = \mathrm{pr}_w^*(\prod_{\alpha \in R(w)} \alpha^\vee)_! \sigma_w^* \mathcal{L}_\psi[2\ell(w)](\ell(w)).$$

As shown in ([11], 2.2.8) this is, up to shift by $\dim X$, an irreducible perverse sheaf on $X(w)$. Finally, one defines $\overline{K(w)}$ to be the Goresky—MacPherson extension of $K(w)$ to the closure $\overline{X(w)}$ of $X(w)$ in $X \times X$. The gluing functors $F_{w,!} : \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_l) \to \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_l)$ are defined by

$$F_{w,!}(A) = p_{2,!}(p_1^*(A) \otimes \overline{K(w)}).$$

In the case when $w = s$ is a simple reflection the morphism $\mathrm{pr}_s : X(s) \to T_s \simeq \mathbb{G}_{m,k}$ extends to $\overline{\mathrm{pr}}_s : \overline{X(s)} \to \mathbb{G}_{a,k}$ and we have

$$\overline{K(s)} \simeq (-\overline{\mathrm{pr}}_s)^* \mathcal{L}_\psi.$$

This leads to the alternative construction of the functor $F_{s,!}$ using the embedding of G/U in a rank-2 vector bundle and the corresponding partial Fourier transform. Namely, let us denote $M_s = \rho_s(\mathrm{SL}_{2,k}) \subset G$ and consider the projection $p_s : X = G/U \to G/Q_s$, where $Q_s = M_s U \subset G$ (note that U normalizes M_s). Now p_s is the complementary open subset to the zero section in a G-equivariant rank-2 symplectic vector bundle $\pi_s : V_s \to G/Q_s$ (see [11]). Furthermore, we have $\overline{X(s)} \simeq X \times_{G/Q_s} X$ and the morphism $-\overline{\mathrm{pr}}_s$ from $\overline{X(s)}$ to $\mathbb{G}_{a,k}$ coincides with the restriction of the symplectic pairing on V_s. It follows that $F_{s,!} = j^* \mathcal{F} j_!$ where $j : X \hookrightarrow V_s$ is the embedding, $\mathcal{F}$ is the (symplectic) Fourier transform for V_s (see the previous section), so $\mathcal{F}^2 \simeq \mathrm{Id}$. Since $j_!$ is right t-exact with respect to perverse t-structures, so is F_s. As

is shown in [11] (see also 5.3 below) for every reduced decomposition $w = s_1 \dots s_l$ one has a canonical isomorphism of functors

$$F_{w,!} \simeq F_{s_1,!} \circ \dots \circ F_{s_l,!}.$$

It follows that all the functors $F_{w,!}$ are right t-exact.

Theorem 5.2.1. *The functors $F_{w,!}$ define a quasi-action of W on $\mathcal{D}_c^b(X, \overline{\mathbb{Q}}_l)$.*

It follows that the functors ${}^pH^0\, F_{w,!}$ can be used to define a W-gluing for $|W|$ copies of $\mathrm{Perv}(X)$. The resulting glued category is denoted by $\mathcal{A}$. This theorem was stated as Theorem 2.6.1 in [11]. However, the proof presented in loc. cit. is insufficient. Namely, it is proved in [11] that the functors $F_{w,!}$ generate the action of the positive braid monoid on $\mathcal{D}_c^b(X, \overline{\mathbb{Q}}_l)$, and the morphisms $F_{s,!}^2 \to \mathrm{Id}$ are constructed. To finish the proof one has to show that the other conditions of Theorem 3.2.2 are satisfied. The condition (1) was checked in 4.2, and the condition (2) will be checked below in 5.5.

5.3. **Isomorphism.** Let us recall from [11] the construction of the canonical isomorphism

$$F_{w,!} \simeq F_{w_1,!} \circ F_{w_2,!}$$

associated with a decomposition $w = w_1 w_2$ such that $\ell(w) = \ell(w_1) + \ell(w_2)$. One starts with a commutative diagram

$$\begin{tikzcd} X(w_1) \times_X X(w_2) \arrow[r] \arrow[d] & T_{w_1} \times T_{w_2} \arrow[d, "m_{w_1,w_2}"] \\ X(w) \arrow[r] & T_w \end{tikzcd} \tag{5.3.1}$$

where the map $m_{w_1,w_2} : T_{w_1} \times T_{w_2} \to T_{w_1 w_2}$ is given by $(t_1, t_2) \mapsto w_2^{-1}(t_1)t_2$. It is easy to check that this diagram is cartesian. Moreover, since $R(w_1 w_2)$ is the disjoint union of $w_2^{-1}(R(w_1))$ and $R(w_2)$, by the base change we get an isomorphism

$$K(w) \simeq p_{13,!}(p_{12}^* K(w_1) \otimes p_{23}^* K(w_2)), \tag{5.3.2}$$

where p_{ij} are the projections from $X(w_1) \times_X X(w_2) \subset X^3$. To derive from this the isomorphism

$$\overline{K(w)} \simeq \overline{K(w_1)} \circ \overline{K(w_2)} \tag{5.3.3}$$

it remains to check that the right-hand side is an irreducible perverse sheaf up to shift by $\dim X$. This is done in [11], (2.4)—(2.5).

5.4. **Geometric information.** We need some more information about varieties $X(w)$. For every $w_1, w_2 \in W$ such that $l(w_1 w_2) = l(w_1) + l(w_2)$ consider the natural maps

$$i_{w_1,w_2} : X(w_1) \times_X X(w_2) \to X(w) \times T_{w_1} : (g_1U, g_2U, g_3U) \mapsto ((g_1U, g_3U), \mathrm{pr}_{w_1}(g_1U, g_2U),$$
$$i'_{w_1,w_2} : X(w_1) \times_X X(w_2) \to X(w) \times T_{w_2} : (g_1U, g_2U, g_3U) \mapsto ((g_1U, g_3U), \mathrm{pr}_{w_2}(g_2U, g_3U),$$

where $w = w_1 w_2$.

Lemma 5.4.1. *The morphism i_{w_1,w_2} (resp. i'_{w_1,w_2}) is an isomorphism onto the locally closed subvariety of pairs $(x, t_1) \in X(w) \times T_{w_1}$ such that $\mathrm{pr}_w(x) t_1^{-1} \in T_{w_2}$ (resp. $(x, t_2) \in X(w) \times T_{w_2}$ such that $\mathrm{pr}_w(x) t_2^{-1} \in w_2^{-1}(T_{w_1})$).*

Proof. The cartesian diagram (5.3.1) allows one to identify $X(w_1) \times_X X(w_2)$ with the subvariety in $X(w) \times T_{w_1} \times T_{w_2}$ consisting of (x, t_1, t_2) such that $\mathrm{pr}_w(x) = w_2^{-1}(t_1)t_2$. Together with the fact that $w_2(t)t^{-1} \in T_{w_2}$ for any $t \in T$, this implies the assertion. $\square$

Let $\mathcal{B} = G/B$ be the flag variety of G, and $O(w) \subset \mathcal{B} \times \mathcal{B}$ the G-orbit corresponding to $w \in W$. The canonical projection $X(w) \to O(w)$ can be considered as a $T \times T_w$-torsor. Namely, we have the natural action of $T \times T$ on $X \times X$ such that $(t, t')(gU, g'U) = (gtU, g't'U)$. Now the subgroup $\{(t, t') \mid t^{-1}t' \in T_w\}$ (which is naturally isomorphic to $T \times T_w$) preserves $X(w)$ and induces the above torsor structure. Let us denote by $\overline{X(w)} \subset X \times X$ and $\overline{O(w)} \subset \mathcal{B} \times \mathcal{B}$ the Zariski closures.

Lemma 5.4.2. *For every $s \in S$ the morphism $\overline{X(s)} \to \overline{O(s)}$ is a $T \times T_s$-torsor.*

This follows essentially from the rank-1 case when $\overline{X(s)} = (\mathbb{A}^2 - 0)^2$, $\overline{O(s)} = \mathbb{P}^1 \times \mathbb{P}^1$.

5.5. **Associativity.** Now we will check the second condition of Theorem 3.2.2 for the functors $F_{w,!}$. Clearly, it is sufficient to check the commutativity of the corresponding diagram of morphisms between kernels on $X \times X$. Let us recall the construction of the morphism

$$c_{s,s} : \overline{K(s)} \circ \overline{K(s)} \to \Delta_* \overline{\mathbb{Q}}_{X,l}.$$

Consider the natural embedding

$$\Delta_s : \overline{X(s)} \hookrightarrow \overline{X(s)} \times_X \overline{X(s)} : (g_1U, g_2U) \mapsto (g_2U, g_1U, g_2U). \tag{5.5.1}$$

We have the morphism

$$\overline{\mathrm{pr}}_{s,s} : \overline{X(s)} \times_X \overline{X(s)} \to \mathbb{G}_{a,k} : (x, x') \mapsto \overline{\mathrm{pr}}_s(x) + \overline{\mathrm{pr}}_s(x')$$

such that $p_{12}^*\overline{K(s)} \otimes p_{23}^*\overline{K(s)} = (-\overline{\mathrm{pr}}_{s,s})^*\mathcal{L}_\psi$. The composition $\overline{\mathrm{pr}}_{s,s}\Delta_s$ is the constant map to $\{0\} \in \mathbb{G}_{a,k}$, hence we get the canonical isomorphism

$$\Delta_s^*(p_{12}^*\overline{K(s)} \otimes p_{23}^*\overline{K(s)}) \simeq \overline{\mathbb{Q}}_{\overline{X(s)},l}[2](1).$$

Now $c_{s,s}$ corresponds by adjunction to the morphism

$$\Delta^*(\overline{K(s)} \circ \overline{K(s)}) \simeq p_{1,!}(\Delta_s^*(p_{12}^*\overline{K(s)} \otimes p_{23}^*\overline{K(s)}) \simeq p_{1,!}(\overline{\mathbb{Q}}_{\overline{X(s)},l}[2](1)) \xrightarrow{\mathrm{tr}} \overline{\mathbb{Q}}_{X,l}$$

where tr is the relative trace morphism for p_1.

Theorem 5.5.1. *Let $sw = ws'$, where $w \in W$, $s, s' \in S$ and $l(sw) = l(w) + 1$. Then the following diagram in $\mathcal{D}_c^b(X \times X, \overline{\mathbb{Q}}_l)$ is commutative:*

$$\begin{tikzcd}
\overline{K(s)} \circ \overline{K(w)} \circ \overline{K(s')} \arrow[r] \arrow[d] & \overline{K(w)} \circ \overline{K(s')} \circ \overline{K(s')} \arrow[d, "c_{s',s'}"] \\
\overline{K(s)} \circ \overline{K(s)} \circ \overline{K(w)} \arrow[r, "c_{s,s}"] & \overline{K(w)}
\end{tikzcd} \tag{5.5.2}$$

where unmarked arrows are induced by the isomorphisms (5.3.3).

The main step in the proof is the following lemma.

Lemma 5.5.2. *Under the conditions of Theorem 5.5.1 there is a canonical isomorphism*

$$\alpha : \overline{X(s)} \times_X X(w) \to X(w) \times_X \overline{X(s')}$$

of schemes over $X \times X$ and a canonical isomorphism

$$\overline{p}_s^*\overline{K(s)} \otimes p_w^*K(w) \simeq \alpha^*(p_w^*K(w) \otimes \overline{p}_{s'}^*\overline{K(s)})$$

which is compatible with the isomorphisms (5.3.2), where e.g. $\overline{p}_s : \overline{X(s)} \times_X X(w) \to \overline{X(s)}$ is the natural projection, etc.

Proof. Consider the morphism

$$\overline{i'}_{s,w} : \overline{X(s)} \times_X X(w) \to X \times X \times T_w : (g_1U, g_2U, g_3U) \mapsto (g_1U, g_3U, \mathrm{pr}_w(g_2U, g_3U))$$

extending the morphism $i'_{s,w}$ defined in 5.4. It is easy to see that $\overline{i'}_{s,w}$ is an embedding. Let us consider the locally closed subvariety $Y(s,w) = O(sw) \cup O(w) \subset \mathcal{B} \times \mathcal{B}$. We have the following isomorphism of $\mathcal{B} \times \mathcal{B}$-schemes:

$$\overline{O(s)} \times_{\mathcal{B}} O(w) \simeq Y(s,w).$$

Using Lemma 5.4.2 we see that the natural projection

$$\overline{p}_{s,w} : \overline{X(s)} \times_X X(w) \to Y(s,w)$$

is a $T \times T_s \times T_w$-torsor, where $T \times T_s \times T_w$ is identified with the subgroup

$$\{(t_1, t_2, t_3) \in T \times T \times T |\ t_1^{-1}t_2 \in T_s, t_2^{-1}t_3 \in T_w\}$$

acting naturally on $\overline{X(s)} \times_X X(w)$. Let $p : X \times X \times T_w \to \mathcal{B} \times \mathcal{B}$ be the projection. Then p is a $T \times T \times T_w$-torsor, and via $\overline{i'}_{s,w}$ we can identify the morphism

$$p^{-1}(Y(s,w)) \to Y(s,w)$$

with the $T \times T \times T_w$-torsor over $Y(s,w)$ induced from $\overline{p}_{s,w}$ by the embedding $T \times T_s \times T_w \hookrightarrow T \times T \times T_w$. In particular, the image of $\overline{i'}_{s,w}$ is a closed subvariety in $p^{-1}(Y(s,w))$. Thus, $\mathrm{im}(\overline{i'}_{s,w})$ is the closure of $\mathrm{im}(i'_{s,w})$ in $p^{-1}(Y(s,w))$.

Similarly, we have an embedding

$$\overline{i}_{w,s'} : X(w) \times_X \overline{X(s)} \to X \times X \times T_w$$

extending $i_{w,s'}$, such that $\mathrm{im}(\overline{i}_{w,s'})$ is the closure of $\mathrm{im}(i_{w,s'})$ in $p^{-1}(Y(w,s'))$, where $Y(ws') = O(ws') \cup O(w)$. Now since $sw = ws'$ we have $Y(s,w) = Y(w,s')$ and we claim that $\mathrm{im}(i'_{s,w}) = \mathrm{im}(i_{w,s'})$, which implies immediately that $\mathrm{im}(\overline{i'}_{s,w}) = \mathrm{im}(\overline{i}_{w,s'})$. Indeed, according to Lemma 5.4.1 the image of $i'_{s,w}$ consists of pairs $(x,t) \in X(sw) \times T_w$ such that $\mathrm{pr}_{sw}(x)t^{-1} \in w^{-1}(T_s)$, while the image of $i_{w,s'}$ consists of (x,t) such that $\mathrm{pr}_{sw}(x)t^{-1} \in T'_s$. Now the equality $w^{-1}sw = s'$ implies that $w^{-1}(T_s) = T_{s'}$ which proves our claim.

Let

$$\alpha : \overline{X(s)} \times_X X(w) \to X(w) \times_X \overline{X(s')}$$

be the unique isomorphism compatible with embeddings $i'_{s,w}$ and $i_{w,s'}$. It remains to check that the sheaves $\overline{p}_s^*\overline{K(s)} \otimes p_w^*K(w)$ and $p_w^*K(w) \otimes \overline{p}_{s'}^*\overline{K(s')}$ correspond to each other under α. Since $K(w)$ is the inverse image of a sheaf on T_w it is sufficient to check that the sheaves $\overline{p}_s^*\overline{K(s)}$ and $\overline{p}_{s'}^*\overline{K(s')}$ correspond to each other under α. Since both these sheaves are local systems (up to shift) it is sufficient to check that $p_s^*K(s)$ and $p_{s'}^*K(s')$ correspond to each other under the restriction of α to the open subset $X(s) \times_X X(w)$. It is easy to check that the following diagram is commutative

$$\begin{tikzcd} X(s) \times_X X(w) \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_s"] & X(w) \times_X X(s') \arrow[d, "\mathrm{pr}'_s"] \\ T_s \arrow[r, "w^{-1}"] & T_{s'} \end{tikzcd} \tag{5.5.3}$$

Moreover, since $l(sw) = l(w) + 1$ we have $w^{-1}(\alpha_s^\vee) = \alpha_{s'}^\vee$, hence the bottom arrow becomes the identity under the identification of T_s (resp. $T_{s'}$) with $\mathbb{G}_m$ via $\alpha_s^\vee$ (resp. $\alpha_{s'}^\vee$), and our assertrion follows immediately. $\square$

Lemma 5.5.3. *Let Y be a scheme, A a correspondence over $Y\times Y$, C and C' symmetric correspondences over $Y\times Y$ such that an isomorphism of $Y\times Y$-schemes is given by*

$$\alpha : C\times_Y A \tilde{\to} A\times_Y C'.$$

Let

$$\Delta_C : C \hookrightarrow C\times_Y C : (y_1,y_2)\mapsto (y_2,y_1,y_2),$$
$$\Delta_{C'}\sigma : C' \hookrightarrow C'\times_Y C' : (y_1,y_2)\mapsto (y_1,y_2,y_1)$$

be the natural embeddings (where $\sigma : C'\to C'$ is the permutation of factors in $Y\times Y$). Then the following diagram is commutative:

$$\begin{tikzcd} C\times_Y A \arrow[r, "\alpha'"] \arrow[d, "\Delta_C"] & A\times_Y C' \arrow[d, "\Delta_{C'}\sigma"] \\ C\times_Y C\times_Y A \arrow[r] & A\times_Y C'\times_Y C' \end{tikzcd} \tag{5.5.4}$$

where the bottom arrow is induced by α, α' is an isomorphism given by

$$\alpha'(y_1,y_2,y_3) = (y_2,y_3,y_2'),$$

for $(y_1,y_2,y_3)\in C\times_Y A$ where $\alpha(y_1,y_2,y_3)=(y_1,y_2',y_3)$.

The proof is straightforward. Note that α' commutes with projections to A, and $\alpha^{-1}\circ\alpha'$ is an involution of $C\times_Y A$.

Proof of Theorem 5.5.1. First we note that $\overline{K(s)}\circ\overline{K(w)}\circ\overline{K(s')}\in {}^p\mathcal{D}^{\le \dim X}(X\times X)$. Indeed, the functor $K\mapsto \overline{K(s)}\circ K$ from $\mathcal{D}^b_c(X\times X,\overline{\mathbb{Q}}_l)$ to itself coincides with the functor $\mathcal{F}_!$ of (4.1) for the rank-2 symplectic bundle $V_s\times X\to G/Q_s\times X$, and hence it is right t-exact with respect to the perverse t-structure. Similarly, the functor $K\mapsto K\circ\overline{K(s')}$ is right t-exact, hence our claim follows from the fact that $\overline{K(w)}[\dim X]$ is a perverse sheaf on $X\times X$. It follows that we can replace all objects in the diagram (5.5.2) by their ${}^pH^{\dim X}$. Since $\overline{K(w)}$ is the Goresky—MacPherson extension from $X(w)\subset X\times_k X$, it is sufficient to check the commutativity of the restriction of (5.5.2) to $X(w)$. Applying Lemma 5.5.3 to the correspondences $A=X(w)$, $C=\overline{X(s)}$ and $C'=\overline{X(s')}$ we obtain the commutative diagram

$$\begin{tikzcd} \overline{X(s)}\times_X X(w) \arrow[r, "\alpha'"] \arrow[d, "\Delta_s"] & X(w)\times_X \overline{X(s')} \arrow[d, "\Delta_{s'}\sigma"] \\ \overline{X(s)}\times_X \overline{X(s)}\times_X X(w) \arrow[r] & X(w)\times_X \overline{X(s')}\times_X \overline{X(s')} \end{tikzcd} \tag{5.5.5}$$

From the construction of the morphisms $c_{s,s}$ and $c_{s',s'}$ and Lemma 5.5.2, we see that it is sufficient to prove that the trivializations (up to shift and twist) of $\Delta_s^*(\overline{p}_{s,1}^*\overline{K(s)}\otimes\overline{p}_{s,2}^*\otimes p_w^*K(w))$ and of $\Delta_{s'}^*(\overline{p}_{s',1}^*\overline{K(s')}\otimes \overline{p}_{s',2}\otimes p_w^*K(w))$ are compatible via the above commutative diagram with α' and the isomorphism of Lemma 5.5.2 (here $\overline{p}_{s,1}$ and $\overline{p}_{s,2}$ are the projections onto the first and the second factors $\overline{X(s)}$). To this end we can replace $\overline{X(s)}$, $\overline{X(s')}$, $\overline{K(s)}$ and $\overline{K(s')}$ by $X(s)$, $X(s')$, $K(s)$ and $K(s')$, respectively. Now this follows immediately from the fact that the isomorphism $\alpha : X(s)\times_X X(w)\simeq X(w)\times_X X(s')$ is compatible with the projections to $T_s\times T_w\simeq T_w\times T_{s'}$ (see the proof of Lemma 5.5.2). $\square$

5.6. **Grothendieck group of the glued category.** It is easy to check that every functor $F_{w,!}$ has the right adjoint $F_{w,*}$. Indeed, $F_{w,!}$ is the composition of the functors $F_{s,!}$ corresponding to simple reflections. Now $F_{s,!} = j^* \mathcal{F} j_!$ where $\mathcal{F}$ is the Fourier transform, and j is an open embedding; hence it is left adjoint to $j^* \mathcal{F} j_*$. In fact, it is shown in [11] that $F_{w,*}(A) = p_{2,*}(p_1^* A \otimes \overline{K(w)})$.

The Proposition 3.4.1 combined with Theorem 2.4.3 gives a simple description of the Grothendieck group of the abelian category $\mathcal{A}$ resulting from gluing on G/U. As we have seen in Corollary 4.1.2 the subgroup $K_{sw,w} \subset K_0(\mathrm{Perv}(G/U))$ coincides with the image of the natural (injective) homomorphism $p_s^* : K_0(\mathrm{Perv}(G/Q_s)) \to K_0(\mathrm{Perv}(G/U))$. Hence, we get the following description of $K_0(\mathcal{A})$.

Theorem 5.6.1. *The group $K_0(\mathcal{A})$ is isomorphic to the group*

$$\mathcal{K} = \{(c_w) \in \oplus_{w \in W} K_0(\mathrm{Perv}(G/U)) \mid \phi_s c_w - c_{sw} \in p_s^*(K_0(\mathrm{Perv}(G/Q_s))),\ w \in W, s \in S\},$$

where ϕ_s is an operator on $K_0(\mathrm{Perv}(G/U))$ induced by the partial Fourier transform $F_{s,!}$.

6. Cubic Hecke algebra

6.1. **A property of the Fourier transform.** Let $\pi : V \to S$ be a symplectic rank-2 bundle, let $\mathcal{F} : \mathcal{D}_c^b(V, \overline{\mathbb{Q}}_l) \to \mathcal{D}_c^b(V, \overline{\mathbb{Q}}_l)$ be the corresponding Fourier transform, $j : U \to V$ the complement to the zero section, and $\mathcal{F}_! = j^* \mathcal{F} j_! : \mathcal{D}_c^b(U, \overline{\mathbb{Q}}_l) \to \mathcal{D}_c^b(U, \overline{\mathbb{Q}}_l)$.

Lemma 6.1.1. *There is a canonical isomorphism of functors*

$$\mathcal{F}_! \circ p^* \simeq p^*[1](1) \tag{6.1.1}$$

where $p = \pi \circ j : U \to S$.

Proof. We start with canonical isomorphisms

$$\mathcal{F} \circ s_* \simeq \pi^*[2](1),$$
$$\mathcal{F} \circ \pi^* \simeq s_*[-2](-1)$$

where $s : S \to V$ is the zero section (see [12]). Now applying $\mathcal{F}$ to the exact triangle

$$j_! p^* F \to \pi^* F \to s_* F \to \dots$$

and using the above isomorphisms we obtain the exact triangle on V:

$$\mathcal{F} j_! p^* F \to s_* F[-2](-1) \to \pi^* F[2](1) \dots$$

Restricting to $U \subset V$ we get the required isomorphism. □

6.2. **Cubic relation.** For every scheme Y we denote by $K_0(Y) = K_0(\mathcal{D}_c^b(Y, \overline{\mathbb{Q}}_l))$ the Grothendieck group of the category $\mathcal{D}_c^b(Y, \overline{\mathbb{Q}}_l)$. We define the action of the algebra $\mathbb{Z}[u, u^{-1}]$ (where u is an indeterminate) on $K_0(Y)$ by setting $u \cdot [F] = [F(-1)]$, where $F \mapsto F(1)$ is the Tate twist.

Proposition 6.2.1. *Let $\phi : K_0(U) \to K_0(U)$ be the operator induced by $\mathcal{F}_!$. Then ϕ satisfies the equation*

$$(\phi + u^{-1})(\phi^2 - 1) = 0. \tag{6.2.1}$$

Proof. From the previous lemma we have $(\phi + u^{-1})|_{\mathrm{im}(p^*)} = 0$. On the other hand, from Proposition 4.1.1 we have $\mathrm{im}(\phi^2 - 1) \subset \mathrm{im}(p^*)$, hence the assertion. □

Corollary 6.2.2. *Let $R = \mathbb{Z}[u, u^{-1}, (u^2-1)^{-1}]$. The submodule $K_0(G/Q_s) \otimes_{\mathbb{Z}[u,u^{-1}]} R \subset K_0(G/U) \otimes_{\mathbb{Z}[u,u^{-1}]} R$ coincides with the image of the operator $\phi^2 - 1$ on the latter R-module.*

Let $r : U \to \mathbb{P}(V)$ be the projection to the projectivization of V, $q : \mathbb{P}(V) \to S$ the projection to the base, so that $p = q \circ r$.

Proposition 6.2.3. *The following relation between operators $K_0(\mathbb{P}(V)) \to K_0(U)$ holds:*

$$\phi r^* = r^* - u^{-1} p^* q_! = r^*(\mathrm{id} - u^{-1} q^* q_!). \tag{6.2.2}$$

Proof. By definition we have

$$\mathcal{F}_!(r^*G) = p_{2!}(p_1^* r^* G \otimes \mathcal{L}|_{U^2})[2](1) \simeq p'_{2!}(p_1'^* G \otimes (r \times \mathrm{id}_U)_!(\mathcal{L}|_{U^2}))[2](1)$$

where p_i (resp. p'_i) are the projections of U^2 (resp. $\mathbb{P}(V) \times U$) on its factors. Note that $(r \times \mathrm{id}_U)_!(\mathcal{L}|_{U^2}) \simeq ((r \times \mathrm{id}_V)_!(\mathcal{L}|_{U \times V}))|_{\mathbb{P}(V) \times U}$. To compute the latter sheaf we decompose r as follows: $r = l \circ k$ where $k : U \hookrightarrow I$ is the open embedding into the incidence correspondence $I \subset \mathbb{P}(V) \times V$ ($k : v \mapsto (\langle v \rangle, v)$), $l : I \to \mathbb{P}(V)$ is the projection. Let $s : \mathbb{P}(V) \to \mathbb{P}(V) \times 0 \subset I$ be the zero section. Then since the images of k and s are complementary to each other we have an exact triangle

$$(k \times \mathrm{id}_V)_!(\mathcal{L}|_{U \times V}) \to \widetilde{\mathcal{L}} \to (s \times \mathrm{id}_V)_*(s \times \mathrm{id}_V)^* \widetilde{\mathcal{L}} \to \dots \tag{6.2.3}$$

where $\widetilde{\mathcal{L}}$ is the pullback of $\mathcal{L}$ by the morphism $I \times V \to V \times V$ induced by the projection $I \to V$. Now we have $(l \times \mathrm{id}_V)_!(s_* s^* \widetilde{\mathcal{L}}) \simeq \overline{\mathbb{Q}}_{l, \mathbb{P}(V) \times V}$ and $(l \times \mathrm{id}_V)_!(\widetilde{\mathcal{L}}) \simeq \overline{\mathbb{Q}}_{l,I}[-2](-1)$. Indeed, the latter isomorphism follows from the fact that $(l \times \mathrm{id}_V)_!(\widetilde{\mathcal{L}})$ is supported on $I \subset \mathbb{P}(V) \times V$ and the trivialiaty of $\widetilde{\mathcal{L}}|_{(l \times \mathrm{id}_V)^{-1}(I)}$. Applying the functor $(l \times \mathrm{id}_V)_!$ to the triangle (6.2.3) and using these isomorphisms we obtain the exact triangle

$$(r \times \mathrm{id}_V)_!(\mathcal{L}|_{U \times V}) \to \overline{\mathbb{Q}}_{l,I}[-2](-1) \to \overline{\mathbb{Q}}_{l, \mathbb{P}(V) \times V} \to \dots$$

Note that the restriction of the projection $I \to V$ to $I \cap (\mathbb{P}(V) \times U)$ is an isomorphism. Hence, passing to Grothendieck groups we obtain

$$[\mathcal{F}_! r^* G] = [r^* G] - [p^* q_! G(1)]$$

as required. $\square$

6.3. **Action of the cubic Hecke algebra.** Now we return to the situation of section 5. Let $\mathcal{H}$ be the Hecke algebra defined as the quotient of the group algebra with coefficients in $\mathbb{Z}[u, u^{-1}]$ where u is indeterminate, of the generalized braid group corresponding to (W, S) by the relations $(s+1)(s-u) = 0$, $s \in S$. Recall that there is an action of $\mathcal{H}$ on $K_0(G/B)$ such that $\mathbb{Z}[u, u^{-1}]$ acts in the standard way (using Tate twist) and the action of s is given by the correspondence $O(s) \subset (G/B)^2$: $O(s) = \{(gB, g'B) \mid g^{-1}g' \in BsB\}$. In terms of the projective bundle $q_s : G/B \to G/M_sB$ associated with s we have $T_s = q_s^* q_{s!} - \mathrm{id}$.

Proposition 6.3.1. *The functors $F_{s,!}$, $s \in S$ extend to the action on $K_0(G/U)$ of the cubic Hecke algebra $\mathcal{H}^c$ which is obtained as the quotient of the group algebra $\mathbb{Z}[u, u^{-1}][B]$ of the braid group B corresponding to (W, S) by the relations $(s+u)(s^2-1) = 0$, $s \in S$. This action preserves $K_0(G/B) \subset K_0(G/U)$ and restricts to the standard action of the quadratic Hecke algebra $\mathcal{H}$ on $K_0(G/B)$ via the $\mathbb{Z}[u, u^{-1}]$-linear homomorphism*

$$\mathcal{H}^c \to \mathcal{H} : s \mapsto -s^{-1}.$$

The proof reduces to a simple computation for the Fourier transform on a symplectic bundle of rank 2.

7. Adjoint functors and canonical complexes for W-gluing

7.1. **Adjoint functors for parabolic subgroups.** Let (W, S) be a finite Coxeter group, $J \subset S$ a subset, and $W_J \subset W$ the subgroup generated by simple reflections in J.

We will frequently use the following fact (see [4], IV, Exercise 1.3): every left (or right) W_J-coset contains a unique element of minimal length. Furthermore, an element $w \in W$ is the shortest element in $W_J w$ if and only if $l(sw) = l(w) + 1$ for every $s \in J$, and in this case we have $l(w'w) = l(w') + l(w)$ for every $w' \in W_J$.

In particular, for any coset $W_J x$ and any $w \in W$ there exists the unique element $p_{W_J x}(w) \in W_J x$ such that $n_{W_J x}(w) := w p_{W_J x}(w)^{-1}$ has minimal possible length. Namely, $n_{W_J x}(w)$ is the shortest element in the coset $wx^{-1}W_J$.

Let $(\mathcal{C}_w, w \in W)$ be a collection of abelian categories, $\Phi_w : \mathcal{C}_{w'} \to \mathcal{C}_{ww'}$, a W-gluing data, and $\mathcal{C}(\Phi)$ the corresponding glued category. For every subset $P \subset W$ let us denote by Φ_P the gluing data on categories $\mathcal{C}_w$, $w \in P$, induced by Φ. We have natural restriction functors $j_P^* : \mathcal{C}(\Phi) \to \mathcal{C}(\Phi_P)$.

We claim that for every coset $W_J x \subset W$ the functor $j_{W_J x}^*$ has the left adjoint functor

$$j_{W_J x,!} : \mathcal{C}(\Phi_{W_J x}) \to \mathcal{C}(\Phi).$$

Indeed, let $A = (A_w, w \in W_J x; \alpha_{w,w'})$ be an object of $\mathcal{C}(\Phi_{W_J x})$. Set $j_{W_J x,!} = (A_w, w \in W; \alpha_{w,w'})$ where

$$A_w = \Phi_{n(w)} A_{p(w)},$$

$n(w) := n_{W_J x}(w)$, $p(w) := p_{W_J x}(w)$, and the morphisms $\alpha_{w,w'} : \Phi_w(A_{w'}) \to A_{ww'}$ are defined as follows. Let us write $p(ww') = w_1 p(w')$ with $w_1 \in W_J$. Then we have

$$wn(w') = n(ww')w_1$$

and $l(n(ww')w_1) = l(n(ww')) + l(w_1)$. Hence, we can define $\alpha_{w,w'}$ as the composition

$$\Phi_w(A_{w'}) = \Phi_w \Phi_{n(w')}(A_{p(w')}) \to \Phi_{wn(w')}(A_{p(w')}) \simeq \Phi_{n(ww')}\Phi_{w_1}(A_{p(w')}) \xrightarrow{\Phi_{n(ww')}(\alpha_{w_1,p(w')})} \Phi_{n(ww')}(A_{w_1 p(w')}) = A_{ww'}.$$

One can easily check that $j_{W_J x,!}$ is indeed an object of $\mathcal{C}(\Phi)$.

Proposition 7.1.1. *The functor $j_{W_J x,!}$ is left adjoint to $j_{W_J x}^*$.*

Proof. Let $A = (A_w, w \in W_J x; \alpha_{w,w'})$ be an object of $\mathcal{C}(\Phi_{W_J x})$, and $B = (B_w, w \in W; \beta_{w,w'})$ an object of $\mathcal{C}(\Phi)$. We have an obvious map

$$\mathrm{Hom}_{\mathcal{C}(\Phi)}(j_{W_J x,!}A, B) \to \mathrm{Hom}_{\mathcal{C}(\Phi_{W_J x})}(A, j_{W_J x}^* B). \tag{7.1.1}$$

The inverse map is constructed as follows. Assume that we are given a morphism $f = (f_w, w \in W_J x) : A \to j_{W_J x}^* B$. Then for every $w \in W$ we define the morphism $\widetilde{f}_w : \Phi_{n(w)} A_{p(w)} \to B$ as the composition

$$\Phi_{n(w)} A_{p(w)} \xrightarrow{\Phi_{n(w)}(f_{p(w)})} \Phi_{n(w)} B_{p(w)} \xrightarrow{\beta_{n(w),p(w)}} B_w.$$

It is easy to check that $\widetilde{f} = (\widetilde{f}_w, w \in W)$ is the morphism in $\mathcal{C}(\Phi)$ between $j_{W_J x,!}A$ and B and that the obtained map

$$\mathrm{Hom}_{\mathcal{C}(\Phi_{W_J x})}(A, j_{W_J x}^* B) \to \mathrm{Hom}_{\mathcal{C}(\Phi)}(j_{W_J x,!}A, B) : f \mapsto \widetilde{f}$$

is inverse to (7.1.1). ◻

If we have an inclusion $W_J x \subset P \subset W$, then the restriction functor $j_{W_J x,P}^* : \mathcal{C}(\Phi_P) \to \mathcal{C}(\Phi_{W_J x})$ has the left adjoint

$$j_{W_J x,P;!} := j_P^* j_{W_J x,!} : \mathcal{C}(\Phi_{W_J x}) \to \mathcal{C}(\Phi_P)$$

(this follows from the proof of the above proposition). Moreover, by construction the composition $j_{W_J x,P}^* j_{W_J x,P;!}$ is the identity functor on $\mathcal{C}(\Phi_{W_J x})$. Hence, we can apply Theorem 1.2.1 to conclude that for any subset $P \subset W$ which is a union of subsets of the form $W_J x$, the category $\mathcal{C}(\Phi_P)$ is obtained by gluing from the categories $\mathcal{C}(\Phi_{W_J x})$ for $W_J x \subset P$.

Let $J \subset K \subset S$. Then one has canonical isomorphisms

$$j_{W_J x}^* \simeq j_{W_J x, W_K x}^* \circ j_{W_K x}^*,$$

$$j_{W_J x,!} \simeq j_{W_K x,!} \circ j_{W_J x, W_K x;!}.$$

Also one has the canonical morphism of functors

$$j_{W_J x,!} j_{W_J x}^* \to j_{W_K x,!} j_{W_K x}^*. \tag{7.1.2}$$

Assume that every functor Φ_w has the left derived $L\Phi_w : \mathcal{D}^-(\mathcal{C}'_w) \to \mathcal{D}^-(\mathcal{C}_{ww'})$ and that these functors satisfy $L\Phi_{w_1} \circ L\Phi_{w_2} \simeq L\Phi_{w_1 w_2}$ when $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. The following proposition gives a sufficient condition for the existence of the derived functor for $j_{W_J x,!}$.

Proposition 7.1.2. *Assume that for every $w \in W$ there is a family of objects $\mathcal{R}_w \subset \operatorname{Ob} \mathcal{C}_w$ that are $\Phi_{w'}$-acyclic for every $w' \in W$ and such that every object of $\mathcal{C}_w$ can be covered by an object in $\mathcal{R}_w$. Then the functor $j_{W_J x,!}$ has the left derived $Lj_{W_J x,!} : \mathcal{D}^-(\mathcal{C}(\Phi_{W_J x})) \to \mathcal{D}^-(\mathcal{C}(\Phi))$ which is left adjoint to the restriction functor $j^*_{W_J x} : \mathcal{D}^-(\mathcal{C}(\Phi)) \to \mathcal{D}^-(\mathcal{C}(\Phi_{W_J x}))$. Furthermore, one has an isomorphism of functors*

$$j^*_w \circ Lj_{W_J x,!} \simeq L\Phi_{n(w)} j^*_{p(w)}.$$

Proof. Let $\mathcal{R} \subset \operatorname{Ob} \mathcal{C}(\Phi_{W_J x})$ be the family of objects that are direct sums of objects of the form $j_{y,W_J x;!} R_y$ where $y \in W_J x$, $R_y \in \mathcal{R}_y$. Clearly, every object of $\mathcal{C}(\Phi_{W_J x})$ can be covered by an object in $\mathcal{R}$. We claim that $\mathcal{R}$ is an adapted class of objects for the functor $j_{W_J x,!}$. Indeed, it suffices to prove that for every $w \in W$ the object $\Phi_{p(w)y^{-1}} R_y$ is $\Phi_{n(w)}$-acyclic, where $w = n(w)p(w)$ is the decomposition used in the definition of $j_{W_J x,!}$. Now we use the fact that

$$\ell(wy^{-1}) = \ell(n(w)p(w)y^{-1}) = \ell(n(w)) + \ell(p(w)y^{-1}).$$

Therefore, by our assumption

$$\Phi_{wy^{-1}} R_y = L\Phi_{wy^{-1}} R_y \simeq L\Phi_{n(w)} \circ L\Phi_{p(w)y^{-1}} R_y = L\Phi_{n(w)}(\Phi_{p(w)y^{-1}} R_y)$$

and our claim follows. Thus, the left derived functor for $j_{W_J x,!}$ exists and can be computed using resolutions in $\mathcal{R}$. The remaining assertions can be easily checked using such resolutions. ☐

Note that Theorem 8.1.1 implies that the conditions of the previous proposition are satisfied for gluing on the basic affine space.

7.2. **Canonical complex.** Let us fix a complete order on S: $S = \{s_1, \dots, s_n\}$. Given an object $A \in \mathcal{C}(\Phi)$ we construct a homological coefficient system on the $(n-1)$-simplex Δ_{n-1} with values in $\mathcal{C}(\Phi)$. Namely, to a subset $J = \{i_1 < \ldots < i_k\} \subset [1, n]$ we assign the object

$$A(J) = \oplus j_{W_{S-J} x,!} j^*_{W_{S-J} x} A$$

where the sum is taken over all right W_{S-J}-cosets. For every inclusion $J \subset J'$ we have the canonical morphism $A(J') \to A(J)$ with components (7.1.2). Thus, we can consider the corresponding chain complex

$$C.(A) : C_{n-1} = A([1,n]) \to \ldots \to C_1 = \oplus_{|J|=2} A(J) \to C_0 = \oplus_{|J|=1} A(J) \tag{7.2.1}$$

The sum of adjunction morphisms

$$C_0(A) = \oplus_{|J|=n-1, W_J x \subset W} j_{W_J x,!} j^*_{W_J x} A \to A$$

induces the morphism

$$H_0(C.(A)) \to A. \tag{7.2.2}$$

It turns out that $C.(A)$ is almost a resolution of A. To describe its homology we need to introduce the functor $\iota : \mathcal{C}(\Phi) \to \mathcal{C}(\Phi)$. For $A = (A_w; \alpha_{w,w'})$ we set $\iota A = (\Phi_{w_0} A_{w_0 w}, \widetilde{\alpha}_{w,w'})$ where w_0 is the longest element in W, the morphism $\widetilde{\alpha}_{w,w'} : \Phi_w \Phi_{w_0} A_{w_0 w'} \to \Phi_{w_0} A_{w_0 ww'}$ is equal to the composition

$$\Phi_w \Phi_{w_0} A_{w_0 w'} \simeq \Phi_{w_0} \Phi_{w_0 w w_0} A_{w_0 w'} \xrightarrow{\Phi_{w_0}(\alpha_{w_0 w w_0, w_0 w'})} \Phi_{w_0} A_{w_0 ww'}.$$

Here we used the following identity in the braid group:

$$\tau(w)\tau(w_0) = \tau(w)\tau(w^{-1} w_0)\tau(w_0 w w_0) = \tau(w_0)\tau(w_0 w w_0).$$

Note that for every $y \in W$ we have the natural morphism

$$\alpha_y : \iota A \to j_{y,!} j^*_y A$$

with components

$$\Phi_{wy^{-1}}(\alpha_{yw^{-1}w_0,w_0w}) : \Phi_{w_0}A_{w_0w} \to \Phi_{wy^{-1}}A_y.$$

It is easy to see that one has the canonical morphism

$$\iota A \to H_{n-1}(C.(A)), \tag{7.2.3}$$

induced by the morphism

$$\iota A \xrightarrow{((-1)^{l(y)}\alpha_y)} C_{n-1}(A) = \oplus_{y\in W} j_{y,!}j_y^* A.$$

Theorem 7.2.1. *One has $H_i(C.(A)) = 0$ for $i \neq 0, n-1$, $H_0(C.(A)) \simeq A$, and $H_{n-1}(C.(A)) \simeq \iota A$.*

Proof. Let us consider the complex $\widetilde{C}.(A)$ obtained from $C.$ by adding the terms $\widetilde{C}_n = \iota A$ and $\widetilde{C}_{-1} = A$ with additional differentials induced by (7.2.2) and (7.2.3). By definition for every $w \in W$ the complex $j_w^*\widetilde{C}.(A)$ looks as follows:

$$\Phi_{w_0}A_{w_0w} \to \oplus_{x\in W}\Phi_{wx^{-1}}A_x \to \oplus_{|J|=1, x\in W_J\backslash W}\Phi_{n_{W_Jx}(w)}A_{p_{W_Jx}(w)} \to \dots$$
$$\to \oplus_{|J|=n-1, x\in W_J\backslash W}\Phi_{n_{W_Jx}(w)}A_{p_{W_Jx}(w)} \to A_w.$$

Recall that as x runs through the set of cosets $W_J\backslash W$, the element $n_{W_Jx}(w)$ runs through all the elements $y \in W$ such that $l(ys) = l(y)+1$ for all $s \in J$. Let us denote by $P_j \subset W$ the set of all $y \in W$ such that $l(ys_j) = l(y)+1$. For $J \subset [1,n]$ we denote $P_J = \cap_{j\in J}P_j$. Then we can rewrite the complex $j_w^*\widetilde{C}.(A)$ as

$$\Phi_{w_0}A_{w_0w} \to \oplus_{y\in W}\Phi_y A_{y^{-1}w} \to \oplus_{j, y\in P_j}\Phi_y A_{y^{-1}w} \to \dots$$
$$\to \oplus_{|J|=n-1, y\in P_J}\Phi_y A_{y^{-1}w} \to A_w.$$

Consider the increasing filtration $F_0 \subset F_1 \subset \dots$ on $j_w^*\widetilde{C}.(A)$ such that F_n contains only summands $\Phi_y A_{y^{-1}w}$ with $l(y) \leq n$. Then the differentials in $\operatorname{gr}_F j_w^*\widetilde{C}.(A)$ are only $\pm\operatorname{id}$ or zeroes. Note that $P_S = \{1\}$ while $\cup_{j=1}^n P_j = W - \{w_0\}$. Hence the complex $\operatorname{gr}_F j_w^*\widetilde{C}.(A)$ is acyclic, and so is $\widetilde{C}.(A)$. ☐

Note that we can attach the end of the complex $C.(\iota A)$ to the beginning of the complex $C.(A)$ via the map

$$C_0(\iota A) \to \iota A \to C_{n-1}(A)$$

to get the complex $\widetilde{C} = \widetilde{C}.(A)$ with homologies $H_0(\widetilde{C}) = A$ and $H_{2n-1}(\widetilde{C}) = \iota^2(A)$ (all other homologies vahish). The functor ι^2 sends the object $A = (A_w, \alpha_{w,w'}) \in \mathcal{C}(\Phi)$ to the object $(\Phi_\pi A_w, \Phi_\pi(\alpha_{w,w'}))$ where $\pi = \tau(w_0)^2 \in B$ is the canonical central element.

The importance of the above construction is that the members of the complex $C.$ are direct sums of objects of the form $j_{W_Jx,!}(\cdot)$ where $J \subset S$ is a proper subset. Hence, it can be used for the induction process. For example, suppose we know that for all proper subsets $J \subset S$ the categories $\mathcal{C}(\Phi_{W_Jx})$ have finite cohomological dimension (this is true for the gluing on the basic affine space if the rank of G is equal to 2). The derived category version of the above construction gives a canonical morphism of functors $\operatorname{Id} \to \Phi_\pi[2n]$. Now an object $A \in \mathcal{C}(\Phi)$ has finite projective dimension if and only if some power of the morphism $A \to \Phi_\pi^k(A)[2nk]$ vanishes.

7.3. **Canonical complex for a "half" of W.** Let us fix an element $s_i \in S$. Recall that we denote by P_i the subset of $w \in W$ such that $l(ws_i) > l(w)$. One has a decomposition of W into the disjoint union of P_i and P_is_i. Let us consider gluing data Φ_{P_i} and $\Phi_{P_is_i}$ corresponding to these two pieces. We are going to construct an analogue of the complex $C.$ for these partial gluing data. As before, for every $A \in \mathcal{C}(\Phi_{P_i})$ we can define a homological coefficient system on Δ_{n-1} with values in $\mathcal{C}(\Phi_{P_i})$ as follows. For every $j \in [1,n]$ let us denote $W^{(j)} = W_{[1,n]-j}$. Now to a nonempty subset $J = \{i_1 < \dots < i_k\} \subset [1,n]$ our coefficient system assigns the object

$$A(J) = \oplus j_{W_{S-J}x, P_i;!}j^*_{W_{S-J}x,P_i}A$$

where the sum is taken over all $x \in W_{S-J} \backslash W$ such that $W^{(j)}x \subset P_i$ for every $j \in J$. For every inclusion $J \subset J'$ we have the canonical morphism $A(J') \to A(J)$. Let us consider the corresponding chain complex

$$C.(P_i, A) : C_{n-1} = A([1, n]) \to \ldots \to C_1 = \oplus_{|J|=2} A(J) \to C_0 = \oplus_{|J|=1} A(J). \qquad (7.3.1)$$

Theorem 7.3.1. *One has $H_0(C.(P_i, A)) \simeq A$ and $H_j(C.(P_i, A)) = 0$ for $j \neq 0$.*

The proof of this theorem will be given in 7.6.

7.4. **Homological lemma.** Let T be a finite set, and (T_i^1) and (T_i^2) two families of subsets of T indexed by $i \in [1, n]$. Assume that we have a family (B_t) of objects of some additive category indexed by $t \in T$. Then we can construct a homology coefficient system on Δ_{n-1} by setting $B(J) = \oplus_{t \in T(J)} B_t$ where

$$T(J) = \cap_{j \in J} T_j^1 \cap \cap_{j \in \overline{J}} T_j^2,$$

$J \subset [1, n]$ is a subset, $\overline{J}$ is the complementary subset. For $J \subset K$ we have the natural map $B(K) \to B(J)$ which is the following composition of the projection and the embedding:

$$\oplus_{t \in T(K)} B_t \to \oplus_{t \in T(J) \cap T(K)} B_t \to \oplus_{t \in T(J)} B_t.$$

Let $D. = C.(\Delta_{n-1}, B(\cdot))$ be the corresponding chain complex.

Lemma 7.4.1. *Assume that for every $t \in T$ the sets $I^1(t) = \{i \mid t \in T_i^1\}$ and $I^2(t) = \{i \mid t \notin T_i^2\}$ are different. Then $H_i(D.) = 0$ for $i > 0$, $H_0(D.) \simeq B(\emptyset) = \oplus_{t \in T(\emptyset)} B_t$ where $T(\emptyset) = \cap_{j=1}^n T_j^2$.*

Proof. Let $\widetilde{D}.$ be the complex

$$B([1, n]) \to \oplus_{|J|=n-1} B(J) \to \ldots \to \oplus_{|J|=1} B(J) \to B(\emptyset)$$

obtained from $D.$ by attaching one more term $B(\emptyset)$. Then $\widetilde{\mathcal{D}}.$ is the direct sum over $t \in T$ of the complexes $\widetilde{D}(t)$ where $\widetilde{D}_i(t) = \oplus_{J : t \in T(J), |J|=i} B_t$. Note that $t \in T(J)$ if and only $I^2(t) \subset J \subset I^1(t)$. Hence, the condition $I^1(t) \neq I^2(t)$ implies that the complex $\widetilde{D}_i(t)$ is exact. $\square$

7.5. **Convexity.**

Lemma 7.5.1. *Let $W' = W_J \subset W$ for some $J \subset S$, and let $y \in P_i$ be an element. Then $W'y \subset P_i$ if and only if $ys_iy^{-1} \notin W'$.*

Proof. The second condition is equivalent to the requirement that the cosets $W'y$ and $W'ys_i$ are different. This is in turn equivalent to the condition that the double coset $W'y\langle s_i \rangle$, where $\langle s_i \rangle = \{1, s_i\}$, has $2|W'|$ elements. Let y_0 be the shortest element in this double coset (see [4], IV, Exercise 1.3). Then every element in $W'y\langle s_i \rangle$ can be written uniquely in the form $w_1 y_0 w_2$ where $w_1 \in W'$, $w_2 \in \langle s_i \rangle$ and $l(w_1 y_0 w_2) = l(w_1) + l(y_0) + l(w_2)$. Since there are $2|W'|$ such expressions this gives a bijection between $W' \times \langle s_i \rangle$ and $W'y\langle s_i \rangle$. The condition $y \in P_i$ implies that $y = w'y_0$ for some $w' \in W'$, i. e. $W'y = W'y_0$. But for every $w_1 \in W'$ one has $l(w_1 y_0 s_i) = l(w_1 y_0) + 1$, hence $W'y_0 \subset P_i$. $\square$

Lemma 7.5.2. *For every $y \in P_i$ the set of $j \in [1, n]$ such that $W^{(j)}y \subset P_i$ coincides with the set of j such that s_j appears in a reduced decomposition of ys_iy^{-1}.*

Proof. Applying Lemma 7.5.1 to $W' = W^{(j)}$ we obtain that $W^{(j)}y \subset P_i$ if and only if $ys_iy^{-1} \notin W^{(j)}$. The latter condition means that s_j appears in every (or some) reduced decomposition of ys_iy^{-1}. $\square$

Consider the graph with vertices W and edges between w and sw for every $w \in W$, $s \in S$. To give a path from w_1 to w_2 in this graph is the same as giving a decomposition of $w_2 w_1^{-1}$ into a product of simple reflections. The shortest paths, *geodesics*, correspond to reduced decompositions. Let us call a subset $P \subset W$ convex if every geodesic between vertices in P lies entirely in P.

Proposition 7.5.3. *The subset P_i is convex.*

Proof. First let us show that for every pair of elements $y, w \in P_i$ there exists a geodesic from y to w which lies entirely in P_i. Arguing by induction in $l(wy^{-1})$, we see that it is sufficient to check that there exists $s_k \in S$ such that either $s_k y \in P_i$ and $l(wy^{-1}s_k) < l(wy^{-1})$, or $ws_k \in P_i$ and $l(s_k wy^{-1}) < l(wy^{-1})$. Assume that there is no such s_k. Note that by Lemma 7.5.1 the condition $s_k y \in P_i$ is equivalent to $ys_i y^{-1} \neq s_k$. In particular, this condition fails to be true for at most one k. Thus, for $w \neq y$ our assumption implies that $ys_i y^{-1} = s_k$ for some k, $ws_i w^{-1} = s_l$ for some l and $l(wy^{-1}s_j) > l(wy^{-1})$ for all $j \neq k$, $l(s_j wy^{-1}) > l(wy^{-1})$ for all $j \neq l$. Let us set $x = wy^{-1} \in W$. Then $xs_k x^{-1} = s_l$ and x is the shortest element in $W^{(l)}xW^{(k)}$. Using Lemma 3.3.2 one can easily see that this is impossible.

Note that every two geodesics with common ends are connected by a sequence of transformations of the following kind: replace the segment

$$y \to s_j y \to s_k s_j y \to \ldots \to w_0(s_j, s_k)y \tag{7.5.1}$$

by the segment

$$y \to s_k y \to s_j s_k y \to \ldots \to w_0(s_j, s_k)y \tag{7.5.2}$$

where $w_0(s_i, s_j)$ is the longest element in the subgroup generated by s_i and s_j. Thus, it is sufficient to check that if segment (7.5.1) lies in P_i, then the corresponding segment (7.5.2) does as well. Applying Lemma 7.5.1 we see that for $y \in P_i$ the segment (7.5.1) lies in P_i if and only if

$$ys_i y^{-1} \notin \{s_j, s_k s_j s_k, s_j s_k s_j s_k s_j, \ldots\}. \tag{7.5.3}$$

Now if $l(w_0(s_i, s_j))$ is odd, then the latter set remains the same if we switch s_j and s_k. Otherwise (if $l(w_0(s_i, s_j))$ is even) the path inverse to segment (7.5.2) has the form

$$w_0(s_j, s_k) \to s_j w_0(s_j, s_k) \to \ldots \sigma_k y \to y.$$

Hence, the conditions $w_0(s_j, s_k) \in P_i$ and (7.5.3) imply that this segment lies in P_i. $\square$

7.6. **Proof of Theorem 7.3.1.** Let us denote by $Q_{i,j}$ the set of $y \in P_i$ such that $W^{(j)}y \subset P_i$. Then for any $w \in P_i$ the complex $j_w^* C.(P_i, A)$ can be written as

$$\oplus_{x \in Q(w,[1,n])} \Phi_{wx^{-1}} A_x \to \ldots \to \oplus_{|J|=2, x \in Q(w,J)} \Phi_{wx^{-1}} A_x \to \oplus_{|J|=1, x \in Q(w,J)} \Phi_{wx^{-1}} A_x \tag{7.6.1}$$

where $Q(w, J) = \cap_{j \in J} Q_{i,j} \cap \cap_{k \in \overline{J}} P_k^{-1} w$, $A_x = j_x^* A$. We can filter this complex by the length of wx^{-1} as in the Proof of Theorem 7.2.1. Now the associated graded factor $\mathrm{gr}_F\, j_w^* C.(P_i, A)$ is the complex of Lemma 7.4.1 with $T = P_i$, $T_j^1 = Q_{i,j}$ and $T_j^2 = P_i \cap P_j^{-1} w$. Since $\cap_{j=1}^n P_k^{-1} w = \{w\}$ it remains to show that the conditions of Lemma 7.4.1 are satisfied in our situation. Indeed, assume that the set of j such that $W^{(j)}y \subset P_i$ coincides with the set of j such that $wy^{-1} \notin P_j$. Let us denote this set by $S_1 \subset [1, n]$. Then for every $j \in S_1$ we have $l(wy^{-1}s_j) < l(wy^{-1})$. Therefore, $l(w_0 wy^{-1}s_j) > l(w_0 wy^{-1})$ for $j \in S_1$, where $w_0 \in W$ is the longest element. This means that $w_0 wy^{-1}$ is the shortest element in the coset $w_0 wy^{-1} W_{S_1}$. Hence, $l(w_0 wy^{-1} w_1) = l(w_0 wy^{-1}) + l(w_1)$ for every $w_1 \in W_{S_1}$. This can be rewritten as $l(wy^{-1}w_1) = l(wy^{-1}) - l(w_1)$ for every $w_1 \in W_{S_1}$. In other words, for every $w_1 \in W_{S_1}$ there exists a geodisic from y to w passing through $w_1 y$. According to Lemma 7.5.3 this implies that $w_1 y \in P_i$ for every $w_1 \in W_{S_1}$. Now by Lemma 7.5.2 we have $ys_i y^{-1} \in W_{S_1}$. Thus, taking $w_1 = ys_i y^{-1}$ we obtain that $ys_i \in P_i$ — contradiction. $\square$

7.7. **Comparison with Beilinson—Drinfeld gluing.** For every nonempty subset $J \subset [1, n]$ let us denote $\mathcal{C}_J = \oplus_{\mathcal{C}}(\Phi_{W_{S-J}x})$ where the sum is taken over $x \in W_{S-J}x \backslash W$ such that $W^{(j)}x \subset P_i$ for every $j \in J$. Then $(\mathcal{C}_J)$ is a family of abelian categories cofibered over the category of nonempty subsets $J \subset [1, n]$ and their embeddings. Namely, if $J \subset K \subset [1, n]$, then we have obvious restriction functors $j^*_{J,K} : \mathcal{C}_J \to \mathcal{C}_K$. Following Beilinson and Drinfeld [2] we denote by $\mathcal{C}_{\rm tot}$ the category of cocartesian sections of $(\mathcal{C}_J)$. An object of $\mathcal{C}_{\rm tot}$ is a collection of objects $A_J \in \mathcal{C}_J$ and isomorphisms $\alpha_{J,K} : A_K \widetilde{\to} j^*_{J,K} A_J$ for $J \subset K$, such that for $J \subset K \subset L$ one has

$$\alpha_{J,L} = j^*_{K,L}\alpha_{J,K} \circ \alpha_{K,L}. \tag{7.7.1}$$

For every $J \subset [1, n]$ we have an obvious restriction functor $j^*_J : \mathcal{C}(\Phi_{P_i}) \to \mathcal{C}_J$ and these functors fit together into the functor $(j^*_\cdot) : \mathcal{C}(\Phi_{P_i}) \to \mathcal{C}_{\rm tot}$.

Theorem 7.7.1. *The functor $(j^*_\cdot) : \mathcal{C}(\Phi_{P_i}) \to \mathcal{C}_{\rm tot}$ is an equivalence of categories.*

Proof. For every $J \subset [1, n]$ we have the left adjoint functor $j_{J,!} : \mathcal{C}_J \to \mathcal{C}(\Phi_{P_i})$ to j^*_J. Namely, $j_{J,!}$ is equal to $j_{W_Jx,P_i;!}$ on $\mathcal{C}(\Phi_{W_{S-J}x}) \subset \mathcal{C}_J$. Now for every $(A_J, \alpha_{J,K}) \in \mathcal{C}_{\rm tot}$ the objects $j_{J,!}A_J$ form a homological coefficient system on Δ_{n-1}, so we can consider the corresponding chain complex $\mathcal{C}.(A.)$. We claim that the functor

$$(A_J) \mapsto H_0(\mathcal{C}.(A.))$$

is quasi-inverse to $(j^*_\cdot)$. Notice that for $A \in \mathcal{C}(\Phi_{P_i})$ the complex $\mathcal{C}.(j^*_\cdot A)$ coincides with the complex (7.3.1), hence by Theorem 7.3.1 it is a resolution of A. It remains to show that for $(A.) \in \mathcal{C}_{\rm tot}$ and for every $J \subset [1, n]$ there is a system of compatible isomorphisms

$$H_0(j^*_J \mathcal{C}.(A.)) \simeq A_J.$$

It is sufficient to construct canonical isomorphisms $\alpha_k : H_0(j^*_{\{k\}}(\mathcal{C}.(A.))) \widetilde{\to} A_{\{k\}}$ for every $k \in [1, n]$. These morphisms are induced by the canonical projections $\mathcal{C}_0(A.) \to A_{\{k\}}$. To check that α_k are isomorphisms it is sufficient to restrict everything by some functor j^*_w and to apply arguments from the proof of Theorem 7.3.1. □

7.8. **Cohomological dimension.** Following [2] let us consider the category $\sec_- = \sec_-(\mathcal{C}.)$ whose objects are collections $A_J \in \mathcal{C}_J$, $J \subset [1, n]$, $\alpha_{J,K} : A_K \to j^*_{J,K} A_J$ for $J \subset K$ satisfying (7.7.1). We consider $\mathcal{C}_{\rm tot}$ as the subcategory in $\sec_-$ and denote by $\mathcal{D}^b_{\rm tot}$ the full subcategory in the derived category $\mathcal{D}^b(\sec_-)$ consisting of complexes $C^\cdot$ with $H^i(C^\cdot) \in \mathcal{C}_{\rm tot}$. The standard t-structure on $\mathcal{D}^b(\sec_-)$ induces a t-structure on $\mathcal{D}^b_{\rm tot}$ with core $\mathcal{C}_{\rm tot}$. As shown in [2] for every $M, N \in \mathcal{C}_{\rm tot}$ there is a spectral sequence converging to $\mathrm{Ext}^{p+q}_{\mathcal{D}_{\rm tot}}(M, N)$ with $E_1^{p,q} = \oplus_{|J|=p+1} \mathrm{Ext}^q_{\mathcal{C}_J}(M_J, N_J)$.

Theorem 7.8.1. *Assume that every object of $\mathcal{C}_w$ can be covered by an object which is acyclic with respect to all the functors $\Phi_{w'}$. Then for every $M, N \in \mathcal{C}_{\rm tot}$ the natural map $\mathrm{Ext}^i_{\mathcal{C}_{\rm tot}}(M, N) \to \mathrm{Ext}^i_{\mathcal{D}_{\rm tot}}(M, N)$ is an isomorphism.*

Proof. It is sufficient to prove that for every element $e \in \mathrm{Ext}^i_{\mathcal{D}_{\rm tot}}(M, N)$ there exists a surjection $M' \to M$ in $\mathcal{C}_{\rm tot}$ such that the corresponding homomorphism $\mathrm{Ext}^i_{\mathcal{D}_{\rm tot}}(M, N) \to \mathrm{Ext}^i_{\mathcal{D}_{\rm tot}}(M', N)$ sends e to zero. Note that for every $w \in P_i$ the functor

$$j_{w,P_i;!} : \mathcal{C}_w \to \mathcal{C}(\Phi_{P_i}) \simeq \mathcal{C}_{\rm tot} \to \sec_-$$

is left adjoint to the restriction functor $j^*_w : \sec_- \to \mathcal{C}_w$. Furthermore, if an object $P \in \mathcal{C}_w$ is acyclic with respect to all the functors $\Phi_{w'}$, then it is also $j_{w,P_i;!}$-acyclic. Thus, for such an object P we have $\mathrm{Ext}^i_{\mathcal{D}_{\rm tot}}(j_{w,P_i;!}P, B) \simeq \mathrm{Ext}^i_{\mathcal{C}_w}(P, j^*_w B)$. Thus, we can start by choosing surjections $P_w \to j^*_w A$ such that P_w is acyclic with respect to all $\Phi_{w'}$ Then for every w we can choose a surjection $P'_w \to P_w$ killing the image of e in the space $\mathrm{Ext}^i_{\mathcal{C}_w}(P_w, j^*_w B)$, and take $M' = \oplus_w j_{w,P_i;!}P'_w$ with the natural morphism $M' \to \oplus_w j_{w,P_i;!} j^*_w M \to M$. □

Corollary 7.8.2. *Assume that the conditions of Theorem 7.8.1 are satisfied and in addition all categories $\mathcal{C}(\Phi_{W_J x})$, where $J \subset [1,n]$ is a proper subset, have finite cohomological dimension (this is true for the gluing on the basic affine space if the rank of G is equal to 2). Then the category $\mathcal{C}(\Phi_{P_i})$ also has finite cohomological dimension.*

7.9. **Gluing from two halves.** Theorem 7.3.1 also implies that the restriction functor $j^*_{P_i} : \mathcal{C}(\Phi) \to \mathcal{C}(\Phi_{P_i})$ has the left adjoint. Namely, for every $A \in \mathcal{C}(\Phi_{P_i})$ we have

$$A = \operatorname{coker}(\oplus_{|J|=2} A(J) \to \oplus_j A(j)).$$

Hence, for any $B \in \mathcal{C}(\Phi)$ we have

$$\operatorname{Hom}(A, j^*_{P_i} B) \simeq \ker(\oplus_j \operatorname{Hom}(A(j), j^*_{P_i} B) \to \oplus_{|J|=2} \operatorname{Hom}(A(J), j^*_{P_i} B)).$$

Now by adjunction we have

$$\operatorname{Hom}(A(J), j^*_{P_i} B) \simeq \oplus \operatorname{Hom}(j^*_{W_{S-J}x, P_i} A, j^*_{W_{S-J}x} B) \simeq \oplus \operatorname{Hom}(j_{W_{S-J}x,!} j^*_{W_{S-J}x,P_i} A, B)$$

where the sum is taken over $x \in W_{S-J} \backslash W$ such that $W^{(j)} x \subset P_i$ for every $j \in J$. It follows that we have an isomorphism

$$\begin{aligned}\operatorname{Hom}(A, j^*_{P_i} B) \simeq & \ker(\operatorname{Hom}(\oplus_{j,x} j_{W^{(j)}x,!} j^*_{W^{(j)}x,P_i} A, B) \to \operatorname{Hom}(\oplus_{|J|=2,x} j_{W_{S-J}x,!} j^*_{W_{S-J}x,P_i} A, B)) \\ \simeq & \operatorname{Hom}(j_{P_i,!} A, B)\end{aligned}$$

where

$$j_{P_i,!} A = \operatorname{coker}(j_{W_{S-J}x,!} j^*_{W_{S-J}x,P_i} A \to \oplus_{j,x} j_{W^{(j)}x,!} j^*_{W^{(j)}x,P_i} A).$$

Theorem 7.3.1 and the above construction work almost literally for the subset $P_i s_i \subset W$ instead of P_i. Now we can consider the functors $j^*_{P_i s_i} j_{P_i,!}$ and $j^*_{P_i} j_{P_i s_i,!}$ as gluing data on the pair of categories $\mathcal{C}(\Phi_{P_i})$ and $\mathcal{C}(\Phi_{P_i s_i})$. Theorem 1.2.1 implies that the corresponding glued category is equivalent to $\mathcal{C}(\Phi)$.

7.10. **Supports of simple objects.** Now we are going to apply the explicit construction of the adjoint functors corresponding to cosets of parabolic subgroups in W to the study of simple objects in the glued category.

Proposition 7.10.1. *Let $P \subset W$ be a subset, S a simple object of $\mathcal{C}(\Phi_P)$, and $x \in P$ an element such that $s_i x \in P$ for some simple reflection s_i. Assume that $S_x = 0$ and $S_{s_i x} \neq 0$. Then $S_w = 0$ for every $w \in P \cap P_i x$.*

Proof. Let $W_i \subset W$ be the subgroup generated by s_i. Our assumptions imply that $j^*_{W_i x, P} S \neq 0$, hence the adjunction morphism

$$A = j_{W_i x, P;!} j^*_{W_i x, P} S \to S$$

is surjective. Now since $S_x = 0$, the explicit construction of $j_{W_i x, P;!}$ tells us that the $A_w = 0$ if w is closer to x than to $s_i x$. The latter condition is equivalent to $wx^{-1} \in P_i$. By surjectivity for such w we also have $S_w = 0$. □

For every object $A = (A_w, w \in W)$ of $\mathcal{C}(\Phi)$ we denote by $\operatorname{Supp}(A)$ the set of $w \in W$ such that $A_w \neq 0$. The above proposition gives serious restrictions on a subset $\operatorname{Supp}(S)$ for a simple object $S \in \mathcal{C}(\Phi)$.

Theorem 7.10.2. *Let S be a simple object of $\mathcal{C}(\Phi)$. Then either $\operatorname{Supp}(S) = W$ or $\operatorname{Supp}(S)$ is an intersection of subsets of the form $P_i x$. In particular, if $\operatorname{Supp}(S) \neq W$, then $\operatorname{Supp}(S)$ is convex.*

Proof. Note that an intersection of subsets of the form $P_i x$ is convex by Lemma 7.5.3. Thus, it suffices to prove that $W - \operatorname{Supp}(S)$ is a union of subsets of the form $P_i x$. Let $w \notin \operatorname{Supp}(S)$. Choose a geodesic path

$$w \to s_{i_1} w \to \dots s_{i_k} \dots s_{i_1} w = x$$

of minimal length from w to an element x of $\operatorname{Supp}(S)$. Then we can apply Proposition 7.10.1 to $x' = s_{i_k} x$ to conclude that $P_{i_k} x' \subset W - \operatorname{Supp}(S)$. On the other hand, clearly $w \in P_{i_k} x'$. □

8.1. **Adapted objects and finiteness of dimensions.** The proof of the following theorem is due to L. Positselski.

Theorem 8.1.1. *Let $k : V \hookrightarrow X$ be an affine open subset such that the projection $G \to X$ splits over V. Then the functors $F_{w,!}k_!$ and $F_{w,*}k_*$ are t-exact.*

Proof. Since $k_!$ is t-exact and $F_{w,!}$ is t-exact from the right, it is sufficient to prove that $F_{w,!}k_!$ is t-exact from the left. Now by definition the functor $F_{w,!}$ is given by the kernel $\overline{K(w)}$ on $X \times X$, hence the functor $F_{w,!}k_!$ is given by the kernel $\overline{K(w)}|_{V\times X}$ on $V \times X$. Since the projection $p_2 : V \times X \to X$ is affine, the functor $p_{2!}$ is left t-exact; hence it is sufficient to prove that for any $A \in \mathrm{Perv}(V)$ the object $p_1^*A \otimes \overline{K(w)}|_{V\times X}$ is a perverse sheaf on $V \times X$. Let $s : V \to G$ be a splitting of the projection $G \to X$ over V. Consider the isomorphism

$$\nu : V \times X \to V \times X : (v, x) \mapsto (v, s(v)x).$$

Then it is sufficient to check that $\nu^*(p_1^*A \otimes \overline{K(w)}) \simeq p_1^*A \otimes \nu^*\overline{K(w)}$ is perverse. The sheaf $\nu^*\overline{K(w)}$ is the Goresky—MacPherson extension of $\nu^*K(w)$ on $\nu^{-1}(X(w))$. By definition

$$\nu^{-1}(X(w)) = \{(v, x) \in V \times X|\ x \in X_w\}$$

where $X_w = Un_wT_wU/U \subset X$ is a locally closed subvariety of X. Note that the projection $\mathrm{pr}_w : X(w) \to T_w$ factors as the composition of projections $p_w : X(w) \to X_w$ and $q_w : X_w \to T_w$ where p_w is smooth of relative dimension $\dim X$, q_w is smooth of relative dimension $l(w)$. Also we have $K(w) = \mathrm{pr}_w^* L_w[l(w)]$ where L_w is a perverse sheaf on T_w. Hence, $K(w) = p_w^*(q_w^*L_w[l(w)])$ where $q_w^*L_w[l(w)]$ is a perverse sheaf on X_w. Now we have $p_w \circ \nu|_{\nu^{-1}(X(w)} = p_2|_{\nu^{-1}(X(w))}$ where $p_2 : V \times X \to X$ is the projection, and hence

$$\nu^*K(w) \simeq p_2^*(q_w^*L_w[l(w)]).$$

It follows that $\nu^*\overline{K(w)} \simeq p_2^*M_w$ where M_w is the Goresky—MacPherson extension of $q_w^*L_w[l(w)]$ to X. Thus,

$$p_1^*A \otimes \nu^*\overline{K(w)} \simeq A \boxtimes M_w,$$

and the latter sheaf is perverse by [3], 4.2.8.

The exactness of functor $F_{w,*}k_*$ follows from the isomorphism

$$F_{w,\psi,*} = D \circ F_{w,\psi^{-1},!} \circ D$$

where D is the Verdier duality (see [11], 2.6.2(i)). □

Theorem 8.1.2. *For any $A, B \in \mathcal{A}$ the spaces $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$ are finite-dimensional.*

Proof. It is sufficient to prove that for every $A \in \mathcal{A}$ there exists a surjection $A' \to A$ in $\mathcal{A}$ such that $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$ is finite-dimensional for every $B \in \mathcal{A}$. We start with the surjection

$$\oplus_w j_{w,!}A_w \to A$$

where $A_w = j_w^*A$. Now we choose a finite covering (U_k) of X by affine open subsets such that the projection $G \to X$ splits over every U_k. Let $j_k : U_k \hookrightarrow X$ be the corresponding open embeddings. Now we replace every A_w by $j_{k,!}j_k^*A_w$ to get the surjection

$$A' = \oplus_{i,w} j_{w,!}j_{k,!}j_k^*A_w \to A.$$

It remains to prove that $\mathrm{Ext}^i_{\mathcal{A}}(j_{w,!}j_{k,!}C, B)$ is finite-dimensional for every $C \in \mathrm{Perv}(U_k)$. According to Theorem 8.1.1 the functor $j_{w,!}j_{k,!} : \mathrm{Perv}(U_k) \to \mathcal{A}$ is exact. Since it is left adjoint to $j_k^*j_w^* : \mathcal{A} \to \mathrm{Perv}(U_k)$ we get the isomorphism

$$\mathrm{Ext}^i_{\mathcal{A}}(j_{w,!}j_{k,!}C, B) \simeq \mathrm{Ext}^i_{\mathrm{Perv}(U_k)}(C, j_k^*j_w^*B),$$

where the latter space is finite-dimensional, as follows from Beilinson's theorem [1]. □

8.2. **Vanishing of Ext^1.** Let (W, S) be a finite Coxeter system, Φ be a W-gluing data.

Theorem 8.2.1. *Assume that for every $w \in W$ and $s \in S$ the following condition holds: for every object $A \in \mathcal{C}_w$ such that the canonical morphism $\Phi_s^2(A) \to A$ is zero, one has $\Phi_s(A) = 0$. Let S and S' be simple objects in $\mathcal{C}(\Phi)$ such that* $\mathrm{Supp}(S) \cap \mathrm{Supp}(S') = \emptyset$. *Then* $\mathrm{Ext}^1_{\mathcal{C}(\Phi)}(S, S') = 0$.

Proof. By assumption we have a partition of W into two subsets I and I' such that $S_w = 0$ for $w \in I'$ and $S'_w = 0$ for $w \in I$. Then we have $E_w = S_w$ for $w \in I$, and $E_w = S'_w$ for $w \in I'$. Now we claim that the structural morphisms $\Phi_s E_w \to E_{sw}$ vanish unless w and sw belong to the same subset of this partition. By the definition of W-gluing this would imply that $E \simeq S \oplus S'$. So assume for example that $w \in I$ and $sw \in I'$. Then the morphism $\Phi_s^2 S_w \to S_w$ is zero, and hence $\Phi_s E_w = \Phi_s S_w = 0$. Similarly, if $w \in I'$ and $sw \in I$, then $\Phi_s E_w = \Phi_s S'_w = 0$. □

Corollary 8.2.2. *Let Φ be a W-gluing data of finite type. Under the assumptions of Theorem 8.2.1, for every pair of simple objects $S = (S_w)$ and $S' = (S'_w)$ of $\mathcal{C}(\Phi)$ the natural map*

$$\mathrm{Ext}^1_{\mathcal{C}(\Phi)}(S, S') \to \oplus_{w \in W} \mathrm{Ext}^1_{\mathcal{C}_w}(S_w, S'_w)$$

is injective.

Proof. If the supports of S and S' do not intersect then we are done by Theorem 8.2.1. So we can assume that there exists $w \in W$ such that S_w and S'_w are both non-zero. Then according to Lemmas 1.3.1 and 1.3.2 we have $S \simeq j_{w,!*} S_w$ and $S' \simeq j_{w,!*} S'_w$. Now assume that we have an extension

$$0 \to S' \to E \to S \to 0 \tag{8.2.1}$$

in $\mathcal{C}(\Phi)$ that induces a trivial extension

$$0 \to S'_w \to E_w \to S_w \to 0. \tag{8.2.2}$$

We claim that the adjunction morphism $j_{w,!} E_w \to E$ is surjective. Indeed, this follows immediately from the commutative diagram with exact rows:

$$\begin{tikzcd} j_{w,!} S'_w \arrow[r] \arrow[d] & j_{w,!} E_w \arrow[r] \arrow[d] & j_{w,!} S_w \arrow[r] \arrow[d] & 0 \\ S' \arrow[r] & E \arrow[r] & S \arrow[r] & 0 \end{tikzcd} \tag{8.2.3}$$

Similarly, one proves that the adjunction morphism $E \to j_{w,*} E_w$ is injective. Thus, $E \simeq j_{w,!*} E_w$, and a splitting of the extension (8.2.2) induces a splitting of (8.2.1). □

Remarks.
1. The conditions of Theorem 8.2.1 are satisfied for the gluing on the basic affine space as one can see combining Corollary 4.1.2 and Lemma 6.1.1.
2. There is an analogue of Theorem 8.2.1 for arbitrary gluing data of finite type. In this case one should impose the condition that $\Phi_{ij}(\mathcal{C}_{ij}) = 0$ (in the notation of section 2). For example, this condition is satisfied for the usual geometric gluing data associated with an open covering (see [11]).

Theorem 8.2.1 can be strengthened using the adjoint functors.

Theorem 8.2.3. *Assume that we have a gluing data Φ as in Theorem 8.2.1. Let $P \subset W$ be a subset such that there exists left and right adjoint functors $j_{P,!}, j_{P,*} : \mathcal{C}(\Phi_P) \to \mathcal{C}(\Phi)$ to the natural restriction functor $j_P^* : \mathcal{C}(\Phi) \to \mathcal{C}(\Phi_P)$. Let S and S' be simple objects in $\mathcal{C}(\Phi)$ such that* $\mathrm{Supp}(S) \cap P \neq \emptyset$, $\mathrm{Supp}(S') \cap P \neq \emptyset$ *but* $\mathrm{Supp}(S) \cap \mathrm{Supp}(S') \cap P = \emptyset$. *Then* $\mathrm{Ext}^1_{\mathcal{C}(\Phi)}(S, S') = 0$.

Proof. The proof of Theorem 8.2.1 shows that there are no non-trivial extensions between $j_P^* S$ and $j_P^* S'$. Now given an extension between S and S' the same argument as in Corollary 8.2.2 shows that it splits. $\square$

For example, let $W = S_3$, Φ be a W-gluing data of finite type. Then the conditions of this theorem are satisfied for any proper connected subgraph $P \subset S_3$. Together with Theorem 7.10.2 this implies that in this case $Ext^1_{\mathcal{C}(\Phi)}(S, S') = 0$ unless either $\mathrm{Supp}(S) \subset \mathrm{Supp}(S')$ or $\mathrm{Supp}(S') \subset \mathrm{Supp}(S)$.

9. Mixed glued category

9.1. **Definition.** We are going to define a mixed analogue of the gluing on the basic affine space. Namely, we consider the situation when the field k is the algebraic closure of $\mathbb{F}_p$ ($p > 2$), and we fix the finite subfield $\mathbb{F}_q \subset k$ such that G, T, and B are defined over $\mathbb{F}_q$. Following [3] we denote by subscript 0 objects defined over this finite field. Thus $X = G/U$ is obtained by extension of scalars from X_0, the corresponding scheme over $\mathbb{F}_q$. Let $\mathrm{Fr} : X \to X$ be the corresponding geometric Frobenius. Since the Fourier transform commutes with Fr, there is a well-defined functor Fr^* on the glued category $\mathcal{A} = \mathcal{A}_\psi$. Let $\mathcal{A}^{\mathrm{Fr}}$ be the category of objects $A \in \mathcal{A}$ together with isomomorphisms $\alpha : A \widetilde{\to} \mathrm{Fr}^* A$. Let $\mathcal{A}_m$ be the subcategory in $\mathcal{A}^{\mathrm{Fr}}$ consisting of $A = (A_w) \in \mathcal{A}$ such that all A_w are mixed perverse sheaves on X_0, with the canonical morphism α. Then $\mathcal{A}_m$ is obtained by gluing from $|W|$ copies of the category $\mathrm{Perv}_m(X_0)$ of mixed perverse sheaves on X_0 (since the relevant functors between perverse sheaves preserve mixedness). We will call an object of $\mathcal{A}_m$ *pure* if all its components are pure perverse sheaves of the same weight.

9.2. **Action of Frobenius on extensions.** Let $\mathcal{C}$ be a $\overline{\mathbb{Q}_l}$-linear abelian category, and $\mathrm{Fr}^* : \mathcal{C} \to \mathcal{C}$ a $\overline{\mathbb{Q}_l}$-linear exact functor. Let $\mathcal{C}^{\mathrm{Fr}}$ be the category of pairs (A, α) where $A \in \mathcal{C}$, $\alpha : A \widetilde{\to} \mathrm{Fr}^* A$ is an isomorphism. Assume that $\mathcal{C}_0$ is a full subcategory of $\mathcal{C}^{\mathrm{Fr}}$. Let us denote by letters with subscript 0 objects in $\mathcal{C}_0$ and omit the subscript when considering the corresponding objects of $\mathcal{C}$. For every pair of objects A_0 and B_0 of $\mathcal{C}_0$ there is a natural automorphism Fr on $\mathrm{Ext}^*_{\mathcal{C}}(A, B)$ induced by Fr^*. Let us assume that for every i the $\overline{\mathbb{Q}_l}$-space $\mathrm{Ext}^i_{\mathcal{C}}(A, B)$ is finite-dimensional and the eigenvalues of the above endomorphism are l-adic units. Finally, notice that if V is a finite-dimensional $\overline{\mathbb{Q}_l}$-vector space with an automorphism ϕ, then we can define a functor of twist with (V, ϕ) on $\mathcal{C}^{\mathrm{Fr}}$ by sending (A, α) to the object $A' = A \otimes_{\overline{\mathbb{Q}_l}} V$ with the isomorphism $\alpha' = \alpha \otimes \phi : A' \widetilde{\to} \mathrm{Fr}^* A'$. Our last assumption is that the subcategory $\mathcal{C}_0 \subset \mathcal{C}^{\mathrm{Fr}}$ is stable under twists by finite-dimensional $\overline{\mathbb{Q}_l}$-vector spaces with continuous $\hat{\mathbb{Z}}$-action.

Theorem 9.2.1. *With the above assumptions the following two conditions are equivalent:*

1. *canonical maps* $\mathrm{Ext}^i_{\mathcal{C}_0}(A_0, B_0) \to \mathrm{Ext}^i_{\mathcal{C}}(A, B)^{\mathrm{Fr}}$ *are surjective for all* $A_0, B_0 \in \mathcal{C}_0$, $i \geq 0$,
2. *for every* $A_0, B_0 \in \mathcal{C}_0$ *and every element* $e \in \mathrm{Ext}^i_{\mathcal{C}}(A, B)$ *there exists a morphism* $f : B_0 \to B'_0$ *such that the image of* e *in* $\mathrm{Ext}^i_{\mathcal{C}}(A, B')$ *is zero.*

Proof. (1) $\Longrightarrow$ (2). Obviously it is sufficient to check the required condition for elements e belonging to one of the generalized eigenspaces of Fr. Twisting A_0 or B_0 with a one-dimensional $\hat{\mathbb{Z}}$-representation we can assume that $(\mathrm{Fr} - 1)^n \cdot e = 0$ for some $n \geq 1$. Consider the element $e_1 = (\mathrm{Fr} - 1)^{n-1} \cdot e$. Then e_1 is invariant under Fr; hence by assumption it is induced by some extension in $\mathcal{C}_0$ between A_0 and B_0. In particular, there exists a morphism $f : B_0 \to B'_0$ such that the image of e_1 in $\mathrm{Ext}^i(A, B')$ is zero. Let e' be the image of e in $\mathrm{Ext}^i(A, B')$. Then $(\mathrm{Fr} - 1)^{n-1} e' = 0$, so we can apply induction to finish the proof.
(2) $\Longrightarrow$ (1). We use induction in i. For $i = 0$ the required surjectivity follows from the assumption that $\mathcal{C}_0$ is a full subcategory of $\mathcal{C}^{\mathrm{Fr}}$. Let $i \geq 1$, $e \in \mathrm{Ext}^i(A, B)$ be an element invariant under Fr. Then by assumption there exists an embedding $B_0 \hookrightarrow B'_0$ killing e. In other words, e is the image of some element $e' \in \mathrm{Ext}^{i-1}(A, B'/B)$ under the boundary map $\delta : \mathrm{Ext}^{i-1}(A, B'/B) \to \mathrm{Ext}^i(A, B)$. Notice that the element $\mathrm{Fr}(e') - e'$ lies in the kernel of δ; hence it comes from some element $d \in \mathrm{Ext}^{i-1}(A, B')$. Applying our assumption to this element we find an embedding $B'_0 \hookrightarrow B''_0$ such that d is killed in $\mathrm{Ext}^{i-1}(A, B'')$. Let e'' be the image of e' in $\mathrm{Ext}^{i-1}(A, B''/B)$. Then e'' is invariant under Fr and maps to e under the

boundary map $\delta' : \mathrm{Ext}^{i-1}(A, B''/B) \to \mathrm{Ext}^i(A, B)$. Now the proof is finished by applying the induction hypothesis to e''. ☐

9.3. **Weights of Ext-groups.** In this section we will temporarily use the notation $\Phi_w = {}^pH^0\, F_{w,!}$ for Kazhdan—Laumon gluing functors (reserving symbols $F.$ for the filtration on mixed objects).

Proposition 9.3.1. *Every simple object of $\mathcal{A}_m$ is pure. Every object A of $\mathcal{A}_m$ has a canonical increasing filtration such that $\mathrm{gr}_n(A)$ is pure of weight n. Every morphism in $\mathcal{A}_m$ is strictly compatible with this filtration.*

Proof. Since $\mathcal{A}_m$ is obtained by gluing, from Lemmas 1.3.1 and 1.3.2 we conclude that every simple object of $\mathcal{A}_m$ has form $j_{w,!*}(A_w)$ for some simple mixed perverse sheaf A_w on X_0. Recall that by [3], 5.3.4 A_w is pure of some weight n. Now since the symplectic Fourier transform preserves weights we obtain that $j_{w,!}(A_w)$ has weights $\le n$, while $j_{w,*}(A_w)$ has weights $\ge n$. Hence, $j_{w,!*}$ is pure of weight n.

Let $A = (A_w)$ be an object of $\mathcal{A}_m$, and let $F.(A_w)$ be the canonical filtration on A_w such that $F_n(A_w)/F_{n-1}(A_w)$ is pure of weight n. Then the weights of $\Phi_s(F_n(A_w))$ are $\le n$; hence the structural morphisms $\Phi_s(A_w) \to A_{sw}$ induce the morphisms $\Phi_s(F_n(A_w)) \to F_n(A_{sw})$. Thus, $(F_n(A_w))$ is a subobject of A for every n, and the $(F.(A_w))$ is the filtration with required properties. ☐

Remark. Another proof of existence of canonical filtrations in $\mathcal{A}_m$ can be obtained using Corollary 8.2.2 and [3], 5.1.15 and 5.3.6.

Theorem 9.3.2. *Let A_0 and B_0 be pure objects in $\mathcal{A}_m$ of weights a and b, respectively. Then weights of Frobenius in $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$ are $\ge i + b - a$.*

Proof. Without loss of generality we can assume that A_0 and B_0 are simple. For $i = 0$ the assertion is clear, since $\mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \oplus_{w\in W} \mathrm{Hom}(j_w^*A, j_w^*B)$ and the weights of Frobenius on the latter space are $\ge b - a$. Similarly, for $i = 1$ the statement follows from Corollary 8.2.2 and [3], 5.1.15. So let us assume that $i > 1$ and that the assertion is true for $i - 1$. Using twist one can easily see that it is sufficient to prove that for $i > a - b$ one has $\mathrm{Ext}^i_{\mathcal{A}}(A, B)^{\mathrm{Fr}} = 0$.

We claim that equivalent conditions of Theorem 9.2.1 are satisfied for the pair of categories $\mathcal{A}_m \subset \mathcal{A}$. Indeed, the action of Frobenius on Ext^i-spaces extends to continuous $\hat{\mathbb{Z}}$-action as follows from the proof of Theorem 8.1.2. Next, the equivalent conditions of 9.2.1 are satisfied for the pair $\mathrm{Perv}_m(X_0) \subset \mathrm{Perv}(X)$ since by Beilinson's theorem the Ext-groups in these categories can be computed in derived categories of construcible sheaves, and then the condition (1) of 9.2.1 follows from "local to global" spectral sequence (see e.g., [3] 5.1). Now we can check the condition (2) of 9.2.1 for $\mathcal{A}_m \subset \mathcal{A}$ as follows. Let $e \in \mathrm{Ext}^i_{\mathcal{A}}(A, B)$ be an element. For every $w \in W$ consider the embedding $j_w^*B_0 \hookrightarrow C_{w,0}$ such that $C_{w,0}$ is $j_{w,*}$-acyclic (such an embedding exists by Theorem 8.1.1). Then $\mathrm{Ext}^i(A, j_{w,*}C_w) \simeq \mathrm{Ext}^i(j_w^*A, C_w)$; hence we can find an embedding $C_{w,0} \to C'_{w,0}$ such that the image of e is killed in $\mathrm{Ext}^i(j_w^*A, C'_w)$. Then

$$B_0 \hookrightarrow B'_0 = \oplus_w j_{w,*}C'_{w,0}$$

is the required embedding. Thus the morphism $\mathrm{Ext}^i_{\mathcal{A}_m}(A, B) \to \mathrm{Ext}^i_{\mathcal{A}}(A, B)^{\mathrm{Fr}}$ is surjective.

Now given an Ioneda extension class in $\mathcal{A}_m$

$$0 \to B_0 \to E_0^1 \to E_0^2 \to \dots E_0^i \to A_0 \to 0 \qquad (9.3.1)$$

we can replace E_0^1 and E_0^2 by their quotients $E_0^1/F_{b-1}(E_0^1)$ and $E_0^2/F_{b-1}(E_0^1)$ (where $F.(\cdot)$ is the weight filtration) to get an equivalent extension class (9.3.1) such that weights of E_0^1 are $\ge b$. Let $C_0 = E_0^1/B_0$. Then the class e of (9.3.1) is an image under the boundary map

$$\mathrm{Ext}^{i-1}(A_0, C_0) \to \mathrm{Ext}^i(A_0, B_0)$$

of some class $e_1 \in \mathrm{Ext}^{i-1}(A_0, C_0)$. Since weights of C_0 are $\geq b$ we have an exact sequence

$$\mathrm{Ext}^{i-1}(A, F_b(C)) \to \mathrm{Ext}^{i-1}(A, C) \to \mathrm{Ext}^{i-1}(A, C/F_b(C)).$$

For every extension class e in $\mathcal{A}_m$ let us denote by $c(e)$ the corresponding extension class in $\mathcal{A}$. By the induction hypothesis the image of $c(e_1)$ in $\mathrm{Ext}^{i-1}(A, C/F_b(C))$ is zero, hence, $c(e_1)$ comes from some class $e_1' \in \mathrm{Ext}^{i-1}(A, F_b(C))$. Now the class $c(e)$ is the image of $c(e_1)$ under the boundary map

$$\mathrm{Ext}^{i-1}(A, C) \to \mathrm{Ext}^{i}(A, B);$$

hence it lies in the image of the boundary map

$$\mathrm{Ext}^{i-1}(A, F_b(C)) \to \mathrm{Ext}^{i}(A, B)$$

corresponding to the exact sequence

$$0 \to B_0 \to F_b(E_0^1) \to F_b(C_0) \to 0.$$

But the image of this exact sequence in $\mathcal{A}$ splits, hence $c(e) = 0$. $\square$

Corollary 9.3.3. *For every $A_0, B_0 \in \mathcal{A}_m$ and every $n \in \mathbb{Z}$ let $\mathrm{Ext}^*_{\mathcal{A}}(A, B)_n$ denotes the weight-n component of $\mathrm{Ext}^*_{\mathcal{A}}(A, B)$. Then all the spaces $\mathrm{Ext}^*_{\mathcal{A}}(A, B)_n$ are finite-dimensional and $\mathrm{Ext}^*_{\mathcal{A}}(A, B)_n = 0$ for $n << 0$.*

10. Induction for representations of braid groups

10.1. **Formulation of the theorem.** Let (W, S) be a finite Coxeter group, B the corresponding braid group, and $B^+ \subset B$ the monoid of positive braids. We fix a subset $J \subset S$ of simple reflections. Let $W_J \subset W$ be the subgroup generated by simple reflections in J. Then (W_J, J) is a Coxeter group and we denote by B_J the corresponding braid group. Let $\mathrm{Mod}_J - B$ be the category of representations of B of the form $\oplus_{x \in W/W_J} V_x$ such that the action of $b \in B$ sends V_x to $V_{\overline{b}x}$ and the following condition is satisfied: for every $s \in S$ and every $x \in W/W_J$ such that $sx \neq x$ one has $s^2|_{V_x} = \mathrm{id}_{V_x}$. Morphisms in $\mathrm{Mod}_J - B$ are morphisms of B-modules preserving direct sum decompositions. Let $\mathrm{Mod}_J - B^+$ be the similar category of B^+-representations. Let $x_0 \in W/W_J$ be the coset containing the identity.

Theorem 10.1.1. *The functor $\oplus_{x \in W/W_J} V_x \mapsto V_{x_0}$ is an equivalence of $\mathrm{Mod}_J - B$ with the category $\mathrm{Mod} - B_J$ of B_J-representations. Similarly the category $\mathrm{Mod}_J - B^+$ is equivalent to $\mathrm{Mod} - B_J^+$.*

10.2. **Arrangements of hyperplanes.** Let us realize W as the group generated by reflections in a Euclidean vector space V over $\mathbb{R}$. Let $\mathcal{H}$ be the corresponding arrangement of hyperplanes in V, X be the complement in $V_{\mathbb{C}}$ to all the complex hyperplanes $H_{\mathbb{C}}$, $H \in \mathcal{H}$. For a subset $J \subset S$ of simple roots we denote by X_J the similar space associated with W_J. By a theorem of P. Deligne [8] the spaces X/W and X_J/W_J are $K(\pi, 1)$ with fundamental groups B and B_J, respectively. Let X'_J be the complement in $V_{\mathbb{C}}$ to all the complex hyperplanes $H_{\mathbb{C}}$ such that the corresponding reflection r_H belongs to W_J. Then X'_J is a W_J-invariant open subset containing X. Furthermore, there is a W_J-equivariant retraction $X'_J \to X_J$ which induces a homotopic equivalence of X'_J/W_J with X_J/W_J. Let us consider the cartesian power $Y_J = (X'_J/W_J)^{W/W_J}$ with the natural action of W by permutations and let us denote by $X \times_W Y_J$ the quotient of $X \times Y_J$ by the diagonal action of W. Let $f : X \times_W Y_J \to X/W$ be the natural projection. Then f is a fibration with fiber Y_J. Let $x_0 \in X$ be a fixed point. Then $y_0 = (x_0, \ldots, x_0) \in Y_J$ is a point fixed by W and hence the map

$$X \to X \times Y_J : x \mapsto (x, y_0)$$

descends to a section $\sigma_0 : X/W \to X \times_W Y_J$ of f. Let $\overline{x_0}$ be the image of x_0 in X/W. Using σ_0 we obtain a canonical identification

$$\pi_1(X \times_W Y_J, \sigma_0(\overline{x_0})) \simeq \pi_1(Y_J, y_0) \rtimes \pi_1(X/W, \overline{x_0}).$$

Note that $\pi_1(X/W, \overline{x_0}) \simeq B$ acts on $\pi_1(Y_J, y_0) \simeq B_J^{W/W_J}$ via the action of W by permutations. In particular, we have the canonical surjection

$$\pi_1(X \times_W Y_J, \sigma_0(\overline{x_0})) \to B_J^{W/W_J} \rtimes W.$$

Now let us consider a W-equivariant map

$$\widetilde{\sigma} : X \to X \times Y_J : x \mapsto (x, \pi_J(\overline{w^{-1}x})_{w \in W/W_J})$$

where $\pi_J : X'_J \to X'_J/W_J$ is the natural projection. Then $\widetilde{\sigma}$ induces a section $\sigma : X/W \to X \times_W Y_J$ of f, and hence a homomorphism

$$\sigma_* : B \to \pi_1(X \times_W Y_J, \sigma(\overline{x_0})).$$

To identify the latter group with $\pi_1(X \times_W Y_J, \sigma_0(\overline{x_0}))$ we have to construct a path from $\sigma_0(x_0)$ to $\sigma(x)$. In other words for every $wW_J \in W/W_J$ we have to construct a path from $\pi_J(x_0)$ to $\pi_J(w^{-1}x_0)$ in X'_J/W_J. To do this we take the canonical representative $w \in W$ of every W_J-coset and consider the corresponding path between x_0 and $w^{-1}x_0$ in X (there is a canonical choice up to homotopy, corresponding to the section $\tau : W \to B = \pi_1(X/W, \overline{x_0})$), then project it to X'_J/W_J. This gives the required identification of the fundamental groups, hence we get a homomorphism

$$f_J : B \to B_J^{W/W_J} \rtimes W.$$

It is easy to check that for every $s \in S$ one has

$$f_J(s) = ((b_w)_{w \in W/W_J}, \overline{s})$$

where w runs over canonical representatives of W/W_J, $b_w = \tau(w^{-1}sw)$ if $w^{-1}sw \in W_J$, and $b_w = 1$ otherwise. Note that since w is a canonical representative the condition $w^{-1}sw \in W_J$ implies that $l(sw) = l(w) + l(w^{-1}sw)$; hence $w^{-1}sw$ is in fact a simple reflection.

10.3. **Proof of Theorem 10.1.1.** The inverse functor $\mathrm{Mod}-B_J \to \mathrm{Mod}_J-B$ is constructed as follows. Let V_0 be a representation of B_J. Then there is a natural action of $B_J^{W/W_J} \rtimes W$ on $V = \oplus_{x \in W/W_J} V_0$ such that B_J^{W/W_J} acts component-wise while W acts by permutations of components in the direct sum. Using the homomorphism f_J we obtain the action of B on V. We claim that V belongs to the subcategory Mod_J-B. Indeed, by the definition of f_J, if $sx \neq x$ for $x \in W/W_J$ and $s \in S$; then s acts by the permutation of factors on $V_x \oplus V_{sx} \subset V$.

It remains to observe that for any $V \in \mathrm{Mod}_J-B$ we can identify all the components V_x with V_{x_0} using the action of the canonical representative for x. This gives an isomorphism of V with the B-representation associated with V_{x_0} as above.

10.4. **Commutator subgroup of the pure braid group.** When J consists of one element we have $B_J = \mathbb{Z}$ and the homomorphism f_J factors through the canonical homomorphism $f : B \to \oplus_{t \in T}\mathbb{Z} \rtimes W$ where T is the set of all reflections in W (=the set of elements that are conjugate to some element of S). By definition $f(s) = (e_{\overline{s}}, \overline{s})$, where e_t is the standard basis of $\oplus_{t \in T}\mathbb{Z}$.

Proposition 10.4.1. *Assume that W is finite. Then f is surjective with the kernel $[P, P]$, the commutator subgroup of the pure braid group $P \subset B$.*

Proof. In the above geometric picture $f|_P$ corresponds to taking the link indices of a loop in X with complex hyperplanes $H_{\mathbb{C}}$ for all $H \in \mathcal{H}$. Thus, the map $f|_P$ can be identified with the natural projection $\pi_1(X) \to H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$. ∎

10.5. **Induction for actions of positive braid monoids.** There is a version of Theorem 10.1.1 concerning the actions of braid monoids (or groups) on categories. Namely, assume that we have a B-action on an additive category $\mathcal{C} = \oplus_{x\in W/W_J}\mathcal{C}_x$ such that the functor $T(b)$ corresponding to $b \in B$ sends $\mathcal{C}_x$ to $\mathcal{C}_{\overline{b}x}$. Assume also that for $s \in S$ and $x \in W/W_J$ such that $sx \neq x$, the functor $T(s) : \mathcal{C}_x \to \mathcal{C}_{sx}$ is an equivalence. Then we can reconstruct $\mathcal{C}$ and the action of B on it from the action of B_J on $\mathcal{C}_{x_0}$ exactly as in Theorem 10.1.1. More precisely, if $w \in W$ is a canonical representative of $x \in W/W_J$, then the functor $T(w) : \mathcal{C}_{x_0} \to \mathcal{C}_x$ is an equivalence, and the corresponding equivalence

$$\mathcal{C} \simeq \oplus_{x\in W/W_J}\mathcal{C}_{x_0}$$

is compatible with the B-actions, where the action of B on the right-hand side is the composition of the natural $B_J^{W/W_J} \rtimes W$-action with f_J.

This is reflected in the following result concerning W-gluing. Let $(\mathcal{C}_w, \Phi)$ be a W-gluing data, and $J \subset S$ a subset of simple reflections. Then it induces a gluing data Φ_J for the categories $(\mathcal{C}_w), w \in W_J$.

Theorem 10.5.1. *Assume that for every $s \in S$ and every $w \in W$ such that $w^{-1}sw \notin W_J$, the morphism $\nu_s : \Phi_s^2|_{\mathcal{C}_w} \to \mathrm{Id}_{\mathcal{C}_w}$ is an isomorphism. Then the glued category $\mathcal{C}(\Phi)$ is equivalent to $\mathcal{C}(\Phi_J)$.*

Proof. Note that our condition on the gluing functors implies that for every $w \in W$, which is the shortest element of wW_J and every $w' \in W_J$, the functors $\Phi_w|_{\mathcal{C}_{w'}}$ and $\Phi_{w^{-1}}|_{\mathcal{C}_{ww'}}$ are quasi-inverse to each other.

We claim that the functors $j^* = j^*_{W_J} : \mathcal{C}(\Phi) \to \mathcal{C}(\Phi_J)$ and $j_! = j_{W_J,!} : \mathcal{C}(\Phi_J) \to \mathcal{C}(\Phi)$ are quasi-inverse to each other. Indeed, we always have $j^*j_! = \mathrm{Id}$. Now let $A = (A_w; \alpha_{w,w'})$ be an object of $\mathcal{C}(\Phi)$. The canonical adjunction morphism $j_!j^*A \to A$ has as components the morphisms

$$\alpha_{n(w),p(w)} : \Phi_{n(w)}A_{p(w)} \to A_w.$$

Since the functors $\Phi_{n(w)}$ and $\Phi_{n(w)^{-1}}$ between $\mathcal{C}_{p(w)}$ and $\mathcal{C}_w$ are quasi-inverse to each other, the associativity condition on α implies that $\alpha_{n(w),p(w)}$ is an isomorphism. ∎

11. Good representations of braid groups

11.1. **Some ideals in the group ring of the pure braid group.** Let (W, S) be a Coxeter group, and B and P the corresponding (generalized) braid group and pure braid groups, respectively. Recall that P is the kernel of the natural homomorphism $B \to W : b \mapsto \overline{b}$. In other words, P is the normal closure of the elements $\{s^2, s \in S\}$ in B. Below we view S as a subset in B and denote by $\overline{s}$ $(s \in S)$ the corresponding elements in W.

Theorem 11.1.1. *There exists a unique collection of right ideals $(I_w, w \in W)$ in $\mathbb{Z}[P]$ such that $I_1 = 0$,*

$$I_{\overline{s}} = (s^2 - 1)\mathbb{Z}[P]$$

for every $s \in S$, and

$$I_{ww'} = I_w + \tau(w)I_{w'}\tau(w)^{-1}$$

for every pair $w, w' \in W$ such that $l(ww') = l(w) + l(w')$.

Proof. We use the induction in $l(w)$, so we assume that all $I(y)$ with $l(y) < l(w)$ are already constructed and satisfy the property

$$I_{yy'} = I_y + \tau(y)I_{y'}\tau(y)^{-1}$$

for $y, y' \in W$ such that $l(yy') = l(y)+l(y') < l(w)$. Choose a decomposition $w = \overline{s}w_1$ with $l(w) = l(w_1)+1$. Then we must have $I_w = I_{\overline{s}} + sI_{w_1}s^{-1}$. It remains to show that the right-hand side does not depend on a choice of decomposition. Let $w = \overline{s'}w_1'$ be another decomposition with $l(w) = l(w_1') + 1$. Assume first that s and s' commute. Then $w_1 = \overline{s'}y$ and $w_1' = \overline{s}y$ where $l(y) = l(w) - 2$, so that by the induction hypothesis we have

$$I_{w_1} = I_{\overline{s'}} + s'I_y(s')^{-1},$$
$$I_{w_1'} = I_{\overline{s}} + sI_ys^{-1}.$$

It follows that

$$I_{\overline{s}} + sI_{w_1}s^{-1} = I_{\overline{s'}} + s'I_{w_1'}(s')^{-1} = I_s + I_{s'} + (ss')I_y(ss')^{-1}.$$

Now if s and s' do not commute there is a defining relation of the form

$$ss's\ldots = s'ss'\ldots$$

where both sides have the same length m. Let us write this relation in the form

$$sr = s'r'$$

where $r = s's\ldots$ and $r' = ss'\ldots$ are the corresponding elements of length $m-1$. In this case we have $w_1 = \overline{r}y$, $w_1' = \overline{r'}y$ where $l(y) = l(w) - m$, so by the induction hypothesis we have

$$I_{w_1} = I_{\overline{r}} + rI_yr^{-1}.$$

Hence

$$I_{\overline{s}} + sI_{w_1}s^{-1} = I_{\overline{s}} + sI_{\overline{r}}s^{-1} + (sr)I_y(sr)^{-1}.$$

Comparing this with the similar expression for $I_{\overline{s'}} + s'I_{w_1'}(s')^{-1}$ we see that it is sufficient to prove the equality

$$I_{\overline{s}} + sI_{\overline{r}}s^{-1} = I_{\overline{s'}} + s'I_{\overline{r'}}(s')^{-1}.$$

In other words, we have to check that

$$I_{\overline{s}} + sI_{\overline{s'}}s^{-1} + ss'I_{\overline{s}}(ss')^{-1} + \ldots = I_{\overline{s'}} + s'I_{\overline{s}}(s')^{-1} + s'sI_{\overline{s'}}(s's)^{-1} + \ldots \tag{11.1.1}$$

where both sides contain m terms. More precisely, we claim that the terms of the right-hand side coincide with those of the left-hand side in inverse order. This follows immediately from the identity $rI_{s''}r^{-1} = I_s$ where $s'' = s$ if m is even, $s'' = s'$ if m is odd. $\square$

Note that I_w is generated by $l(w)$ elements and $I_w \subset I_y$ if $y = ww'$ and $l(ww') = l(w) + l(w')$. In the case when W is finite, this implies that I_{w_0} contains all the ideals I_w, where w_0 is the longest element in W.

Corollary 11.1.2. *Assume that W is finite. Then $I_{w_0} = I$ where I is the augmentation ideal in $\mathbb{Z}[P]$.*

Proof. Let $s \in S$. Then we have

$$I_{w_0} = I_s + sI_{sw_0}s^{-1}.$$

Hence,

$$sI_{w_0}s^{-1} = I_s + s^2I_{sw_0} = I_s + I_{sw_0};$$

here the last equality follows from the definition of I_s. In particular, $sI_{w_0}s^{-1} \subset I_{w_0}$ for every $s \in S$. It follows that

$$s^{-1}I_{w_0}s = s^{-2}(sI_{w_0}s^{-1}) \subset s^{-2}I_{w_0} \subset I_{w_0} + I_s = I_{w_0}.$$

Thus, I_{w_0} is stable under the action of B by conjugation. Since it also contains (s^2-1) for every $s \in S$, it should be equal to I. $\square$

11.2. **Main definition.** Let V be a representation of the braid group B. For every $w \in W$ let us denote

$$V_w = I_wV \subset V$$

where $I_w \subset \mathbb{Z}[P]$ is the ideal defined above. Then $V_s = (s^2-1)V$ for $s \in S$, and

$$V_{w_1w_2} = V_{w_1} + \tau(w_1)V_{w_2}$$

if $l(w_1w_2) = l(w_1) + l(w_2)$. Define $K_W(V)$ to be the following subspace of V^W:

$$K_W(V) = \{(x_w), w \in W : x_w \in V, x_{\overline{s}w} - sx_w \in V_s, \forall s \in S, w \in W\}.$$

Proposition 11.2.1. *For every $(x_w) \in K_W(V)$ and every $w_1, w_2 \in W$, one has $x_{w_1w_2} - \tau(w_1)x_{w_2} \in V_{w_2}$. For every $y \in W$ there is a natural embedding $i_y : V \to K_W(V)$ given by*

$$i_y(v) = (\tau(wy^{-1})v, w \in W).$$

The proof is straightforward.

Example. Let $R = \mathbb{Z}[u, u^{-1}, (u^2-1)^{-1}]$ where u is an indeterminate. It follows from Theorem 5.6.1 and Corollary 6.2.2 that the R-module $K_0(\mathcal{A}) \otimes_{\mathbb{Z}[u,u^{-1}]} R$ is naturally isomorphic to $K_W(K_0(G/U) \otimes_{\mathbb{Z}[u,u^{-1}]} R)$ where $\mathcal{A}$ is the Kazhdan—Laumon category. The embeddings i_y correspond to functors $Lj_{y,!}$ (left adjoint to restrictions).

Definition 11.2.2. A B-representation V is called *good* if $K_W(V)$ is generated by the subspaces $i_y(V)$, $y \in W$.

In the situation of the above example it would be very desirable to prove that an appropriate localization of $K_0(G/U)$ is a good representation of B since this would imply that the objects of finite projective dimension generate the corresponding localization of $K_0(\mathcal{A})$. Then the following result could be applied to construct a bilinear pairing on appropriate finite-dimensional quotient of $K_0(\mathcal{A})$.

Proposition 11.2.3. *Let V be a finite-dimensional representation of B, and V^* the dual representation. Then both V and V^* are good if and only if there exists a non-degenerate pairing $\chi : K_W(V) \otimes K_W(V^*) \to \mathbb{C}$ such that $\chi(i_y(v_y), v') = \langle v_y, p_y v' \rangle$ and $\chi(v, i_y(v'_y)) = \langle p_y v, v'_y \rangle$ for every $y \in W$, $v_y \in V$, $v'_y \in V^*$, $v \in K_W(V)$, $v' \in K_W(V^*)$.*

Proof. If such a pairing χ exists, then the orthogonal to the subspace in V generated by elements of the form $i_y(v_y)$, $y \in W$, $v_y \in V$, is zero, and hence, V is good. Similarly, V^* is good. Now assume that both V and V^* are good. Then we have the surjective map

$$\pi : \oplus_{y \in W} V \to K_W(V) : (v_y, y \in W) \mapsto \sum_y i_y(v_y)$$

and the similar map $\pi' : \oplus_{y \in W} V^* \to K_W(V^*)$. It is easy to see that the kernel of π consists of collections $(v_y, y \in W)$ such that $\sum_{y \in W} \tau(wy^{-1})v_y$ for every $w \in W$. The kernel of π' has a similar description. Now we define a pairing

$$\widetilde{\chi} : (\oplus_{y \in W} V) \otimes (\oplus_{y \in W} V^*) \to \mathbb{C}$$

by the formula

$$\widetilde{\chi}((v_y, y \in W), (v'_y, y \in W)) = \sum_{y,w \in W} \langle v_w, \tau(wy^{-1})v'_y \rangle = \sum_{y,w \in W} \langle \tau(yw^{-1})v_w, v'_y \rangle.$$

Since the left and right kernels of $\widetilde{\chi}$ coincide with $\ker(\pi)$ and $\ker(\pi')$, respectively, it descends to the non-degenerate pairing χ between $K_W(V)$ and $K_W(V^*)$. $\square$

Let Br_n be the Artin braid group such that the corresponding Coxeter group is the symmetric group S_n.

Proposition 11.2.4. *Any representation of $Br_2 = \mathbb{Z}$ is good.*

Proof. A representation of Br_2 is a space V with an operator $\phi : V \to V$. The corresponding space $K_{S_2}(V)$ consists of pairs $(v_1, v_2) \in V^2$ such that $v_2 - \phi v_1 \in (\phi^2 - 1)V$. The maps i_1 and i_s (where s is the generator of Br_2) are given by $i_1(v) = (v, \phi v)$, $i_s(v) = (\phi(v), v)$. Now for any $(v_1, v_2) \in K_{S_2}(V)$ we have

$$(v_1, v_2) = i_1(v_1 + \phi(v)) + i_s(v)$$

where $v \in V$ is such that $v_2 - \phi v_1 = (\phi^2 - 1)v$. $\square$

Theorem 11.2.5. *Let V be a representation of Br_3 over a field k, such that for generators s_1 and s_2, the following identity is satisfied in V:*

$$(s^2-1)(s-\lambda)=0$$

where $\lambda \in k$ is an element such that $\lambda^6 \neq 1$. Then V is good.

11.3. **Criterion.** We need an auxiliary result which allows one to check that a subspace of V^2 is of the form $K_{S_2}(V)$ for some action of $Br_2 = \mathbb{Z}$ on V. It is more natural to generalize this construction as follows. Let $V = V_0 \oplus V_1$ be a super-vector space, and $\phi : V \to V$ an odd operator. Then ϕ^2 is even, and we have the super-subspace $V_\phi = (\phi^2-1)V \subset V$. Now we define the vector space $K(\phi) = \{v \in V | \, \phi(v) - v \in V_\phi\}$. Note that this is a non-homogeneous subspace of V, equipped with two sections i_0 and i_1 of the projections to V_0 and V_1, namely, $i_\alpha(v_\alpha) = v_\alpha + \phi(v_\alpha)$ for $\alpha = 0, 1$. We want to characterize non-homogeneous subspaces of V arising in this way.

Proposition 11.3.1. *Let $K \subset V$ be a non-homogeneous subspace, and $i_0 : V_0 \to K$ and $i_1 : V_1 \to K$ sections of the projections of K to V_0 and V_1. Let K^h be the maximal homogeneous subspace of K. Let $\phi : V \to V$ be the odd operator with components $p_1 i_0$ and $p_0 i_1$ where $p_\alpha : K \to V_\alpha$ are the natural projections. Then we have inclusions $V_\phi \subset K^h$ and $K(\phi) \subset K$. The equality $K = K(\phi)$ holds if and only if $V_\phi = K^h$.*

11.4. **Proof of Theorem 11.2.5.** . We will prove first that $K_{S_3}(V) = K_{S_2}(V')$ where V' is some representation of Br_2. Namely, let V' be the space of triples $(x, y, z) \in V^3$ such that $y - s_1 x \in V_{s_1}$ and $y - s_2 z \in V_{s_2}$. We have the natural embedding

$$\kappa : K_{S_3}(V) \hookrightarrow V' \oplus V' : (x_w, w \in S_3) \mapsto ((x_1, x_{s_1}, x_{s_2 s_1}), (x_{s_2}, x_{s_1 s_2}, x_{s_2 s_1 s_2})).$$

To apply Proposition 11.3.1 to the image of κ we have to define maps $i_0, i_1 : V' \to \kappa K_{S_3}(V)$ such that $p_\alpha i_\alpha = \mathrm{id}$ for $\alpha = 0, 1$. Let us set

$$\begin{aligned} i_0(x,y,z) &= ((x,y,z),(s_2x, s_1s_2x - s_2s_1s_2y + s_2s_1z, s_1z)),\\ i_1(x,y,z) &= ((s_2x, s_1s_2x - s_2s_1s_2y + s_2s_1z, s_1z),(x,y,z)). \end{aligned}$$

Then the operator $\phi = p_1 i_0 = p_0 i_1 : V' \to V'$ has the following form:

$$\phi(x,y,z) = (s_2x, s_1s_2x - s_1s_2s_1y + s_2s_1z, s_1z).$$

Now according to Proposition 11.3.1 in order to check that $\kappa K_{S_3}(V)$ is obtained by gluing from $(V' \oplus V', \phi)$ we have to check that the image of the operator $\phi^2 - \mathrm{id}$ is precisely the subspace $U = \{(x,y,z) \in V' | \, x \in V_{s_2}, z \in V_{s_1}\}$. Let $u \in U$ be an arbitrary element. Then

$$u = (\phi^2 - \mathrm{id})(\lambda^2-1)^{-1}u + u'$$

with $u' = (0, y, 0) \in U$. Then $y \in V_{s_1} \cap V_{s_2}$. Hence, $\tau(w_0)$ acts on y as multiplication by λ^3. It follows that $u' = (\phi^2 - \mathrm{id})(\lambda^6-1)^{-1}u'$. Hence, u' and u are in the image of $\phi^2 - \mathrm{id}$ as required.

It follows from Proposition 11.2.4 that $K_{S_3}(V) = K_{S_2}(V')$ is generated by the images of i_0 and i_1. Thus, we are reduced to show that V' is generated by elements of the form (x, s_1x, s_2s_1x), (s_1y, y, s_2y) and (s_1s_2z, s_2z, z) which is straightforward. □

11.5. **Representations of quadratic Hecke algebras are good.** Let H_q be the Hecke algebra of (W, S) over $\mathbb{C}$ with complex parameter $q \in \mathbb{C}$. Recall that H_q is the quotient of the group algebra $\mathbb{C}[B]$ where B is the corresponding braid group by the quadratic relations $(s-q)(s+1) = 0$ for every $s \in S$.

Theorem 11.5.1. *Assume that q is not a root of unity. Then every representation of H_q is good.*

Corollary 11.5.2. *Let V be a representation of B such that $(s-\lambda)(s-\mu) = 0$ for every $s \in S$, where $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C}$, and $\frac{\mu}{\lambda}$ is not a root of unity. Then V is good.*

Recall that we denote $\pi = \tau(w_0)^2 \in B$ where $w_0 \in W$ is the longest element in W.

Lemma 11.5.3. *Assume that q is not a root of unity. Let V be an irreducible representation of H_q. Then either $\pi - 1$ acts by a non-zero constant on V or V is the one-dimensional representation such that $s = -1$ on V for every $s \in S$.*

Proof. Let E be an irreducible representation of W, $E(u)$ be the corresponding irreducible representation of the Hecke algebra H of (W, S) over $\mathbb{Q}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$ defined by Lusztig. According to [13] (5.12.2) one has

$$\pi = u^{2l(w_0) - a_E - A_E}$$

on $E(u)$ where the integers $a_E \geq 0$ and $A_E \geq 0$ are the lowest and the highest degrees of u appearing with non-zero coefficient in the formal dimension $D_E(u)$ of E. Recall that $D_E(u)$ is defined from the equation

$$D_E(u) \cdot \sum_{w \in W} u^{-l(w)} \operatorname{Tr}(\tau(w), E(u))^2 = \dim(E) \cdot \sum_{w \in W} u^{l(w)}.$$

It is known that $a_{E \otimes sign} = l(w_0) - A_E$ where *sign* is the sign representation of W. In particular, $a_E \leq A_E \leq l(w_0)$. Thus, we have $\pi = u^l$ on $E(u)$, where $l > 0$ unless $a_E = A_E = l(w_0)$. In the latter case $D_E(u) = f_E^{-1} u^{l(w_0)}$ where $f_E > 0$ is an integer. Thus, $D_E(1) = \dim(E) = f_E^{-1} = 1$ and $E = sign$. ☐

Proof of Theorem 11.5.1. Since H_q is finite-dimensional it is sufficient to prove the assertion for a finite-dimensional representation V of H_q. Clearly, we can assume that V is irreducible. Assume first that V is one-dimensional and $s = -1$ on V for every $s \in S$. Then $V_s = 0$ for every s so $K_W(V) \simeq V$ and all the maps i_y are isomorphisms, hence V is good. Thus, according to Lemma 11.5.3 we can assume that $\pi - 1 \neq 0$ on V. Now we are going to use the K_0-analogue of the complex defined in section 7. First of all for every coset $W_J x \in W$ we have the map $i_{W_J x} : K_{W_J}(V) \to K_W(V)$ which is a section of the projection $p_{W_J x}$ onto components $W_J x \subset W$. Namely, using notation of section 7 the w-component of $i_{W_J x}(v_y, y \in W_J x)$ is $n_{W_J x}(w)(v_{p_{W_J x}(w)})$. To prove that V is good it is sufficient to show that $K_W(V)$ is generated by images of all maps $i_{W_J x}$ where $J \subset S$ is a proper subset. Now the proof of Theorem 7.2.1 shows that for every $v \in K_W(V)$, we have the identity

$$\sum_{J \subset S, |J| < n, x \in W_J \backslash W} (-1)^{|J|} i_{W_J x} p_{W_J x}(v) = \iota(v) + (-1)^{n-1} v,$$

where $W_\emptyset = \{1\}$, $\iota(v_w, w \in W) = (w_0 v_{w_0 w}, w \in W)$. One has $\iota^2(v_w, w \in W) = (\pi v_w, w \in W)$. It follows that for every $v \in K_W(V)$ the element $(\pi - 1)v$ is a linear combination of elements of the form $i_{W_J x}(v')$ where $J \subset S$ is a proper subset, and hence v itself is such a linear combination. ☐

11.6. **Good representations and cubic Hecke algebras.** Let H_q^c be the cubic Hecke algebra with complex parameter q, i.e., the quotient of the group algebra $\mathbb{C}[B]$ of the braid group B of (W, S) by the relations $(s - q)(s^2 - 1) = 0$, $s \in S$, where $q \in \mathbb{C}$ is a constant. An optimistic conjecture would be that if q is not a root of unity, then every representation of B that factors through H_q^c is good. The particular cases are Proposition 11.2.4 and Theorems 11.2.5, 11.5.1. Below we check some other particular cases of this conjecture. However, it seems that in general one needs to add some higher polynomial identities on the generators in order for such a statement to be true (see [15] for an example of such identities).

Theorem 11.6.1. *Let (W, S) be of type A_n for $n \leq 3$. Then every representation of H_q^c is good provided that q is not a root of unity.*

Theorem 11.6.2. *Let (W, S) be of type B_2. Then every representation of H_q^c is good provided that $q^8 \neq 1$.*

The structure of the proof of these two theorems is the same as of Theorem 11.2.5. We choose a simple reflection $s_i \in S$ ($s_2 \in S_3$ in Theorem 11.2.5) and consider the partition of W into two pieces: P_i and $P_i s_i$. This way we get an inclusion $K_W(V) \hookrightarrow V_0' \oplus V_1'$ where $V_0' \subset \oplus_{P_i} V$ is the space of collections $(v_w, w \in P_i)$ such that $v_{sw} - sv_w \in V_s$ whenever $w, sw \in P_i$; V_1' is the similar space for $P_i s_i \subset W$ instead of P_i. Let p_0 and p_1 be the corresponding projections of $K_W(V)$ to V_0' and V_1'. We are going to construct sections s_0 and s_1 of these projections. For every coset $W_J x \subset P_i$ (resp. $W_J x \subset P_i s_i$) let us consider the map $i_{W_J x, P_i} = p_0 i_{W_J x} : K_{W_J} \to V_0'$ (resp. $i_{W_J x, P_i s_i} = p_1 i_{W_J x} : K_{W_J} \to V_1'$). The proof of Theorem 7.3.1 shows that for every $v' \in V_0'$ we have the identity

$$\sum_{J,x} (-1)^{|J|-1} i_{W_{S-J}x, P_i} p_{W_{S-J}x}(v') = v',$$

where the sum is taken over non-empty subsets $J \subset S$ and $x \in W_{S-J}\backslash W$ such that $W^{(j)}x \subset P_i$ for every $j \in J$. Now we define s_0 to be the similar sum

$$s_0(v') = \sum_{J,x} (-1)^{|J|-1} i_{W_{S-J}x} p_{W_{S-J}x}(v').$$

Lemma 11.6.3. *For every $y \in P_i$ one has $i_y = s_0 i_{y,P_i}$.*

Proof. Note that for a coset $W_J x \subset P_i$ we have $p_{W_J x} i_{y,P_i} = i_{x', W_J x}$, where x' is the element of $W_J x$ closest to y. Now the proof of Theorem 7.3.1 shows that we have a formal equality

$$\sum_{J,x} (-1)^{|J|-1} x'(W_{S-J}x) = y,$$

where the sum is taken over non-empty subsets $J \subset S$ and $x \in W_{S-J}\backslash W$ such that $W^{(j)}x \subset P_i$ for every $j \in J$, $x'(W_{Kx})$ denotes the element in $W_K x$ closest to y. The assertion follows immediately, since for every $x' \in W_K x$ we have $i_{W_K x} i_{x', W_K x} = i_{x'}$. $\square$

Similarly, one constructs the section $s_1 : V_1' \to K_W(V)$. Thus, it is sufficient to show that the subspace $K_W(V) \subset V_0' \oplus V_1'$ coincides with the subspace $K(\phi)$ where $\phi|_{V_0'} = p_1 s_0 : V_0' \to V_1'$, $\phi|_{V_1'} = p_0 s_1 : V_1' \to V_0'$. By Proposition 11.3.1 this amounts to showing that the image of $\phi^2 - \mathrm{id}$ surjects onto $\overline{V}_0 \oplus \overline{V}_1$, where e.g., $\overline{V}_0 = K_W(V) \cap V_0'$ consists of elements $(v_y, y \in P_i)$ such that $v_y \in V_{ys_iy^{-1}}$ whenever $ys_iy^{-1} \in S$. One way to show it would be to consider some natural filtration on $\overline{V}_0$ preserved by ϕ^2 and considering the induced map on the associated graded factors. The filtration is defined as follows. The subspace $F_k(V_0') \subset \overline{V}_0$ for $k \geq 0$ consists of $(v_y, y \in P_i) \in \overline{V}_0$ such that $v_y = 0$ whenever $l(ys_iy^{-1}) \leq 2k-1$. There is a similar filtration on $\overline{V}_1$ (replace P_i by $P_i s_i$ everywhere). We claim that ϕ is compatible with these filtrations. As we will see below this is a consequence of the definition of s_0 and of the following result.

Proposition 11.6.4. *Let $y, w \in W$ be a pair of elements. Assume that $l(ys_iy^{-1}) > l(ws_iw^{-1})$. Then no geodesics from y to w pass through ys_i.*

Proof. Let us denote $r = ws_iw^{-1}$, $r' = ys_iy^{-1}$, $w_1 = yw^{-1}$. The existence of a geodesic from y to w passing through ys_i is equivalent to the existence of a geodesics from w_1 to 1 passing through $ys_iw^{-1} = w_1 r$. In other words this is equivalent to the equality $l(w_1) = l(w_1 r) + l(w_1 r w_1^{-1})$, i.e., $l(w_1) = l(w_1 r) + l(r')$ (since $w_1 r w_1^{-1} = r'$). But one has $l(w_1) \leq l(w_1 r) + l(r) < l(w_1 r) + l(r')$, which contradicts the above equality. $\square$

Corollary 11.6.5. *The map $\phi : V_0' \to V_1'$ sends $F_k(\overline{V}_0)$ to $F_k(\overline{V}_1)$.*

Proof. As we have seen in the proof of Theorem 7.3.1, in the sum defining $p_w s_0$ all the terms cancel except those that correspond to $x \in P_i$, such that the set $\{s \in S \mid l(wx^{-1}s) < l(wx^{-1})\}$ coincides with the set of s_j such that $W^{(j)}x \subset P_i$. By Lemma 7.5.2 the latter set coincides with the set $S(xs_ix^{-1})$ of simple reflections appearing in any reduced decomposition of xs_ix^{-1}. In particular, this implies that for

any $w_1 \in W_{S(xs_ix^{-1})}$ there exists a geodesic from x to w passing through w_1x. Taking $w_1 = xs_ix^{-1}$ we see that for such x there exists a geodesic from x to w passing though xs_i. According to Proposition 11.6.4 this implies that $l(xs_ix^{-1}) \le l(ws_iw^{-1})$, hence the assertion. □

Lemma 11.6.6. *Let $w \in W$ be an element, $r \in W$ a reflection, and $S(r)$ the set of simple reflections in a reduced decomposition of r. Assume that $l(wrw^{-1}) = l(r)$ and the set $\{s \in S \mid l(ws) < l(w)\}$ coincides with $S(r)$. Assume also that W is either of type A_n or of rank 2. Then w is the longest element of $W_{S(r)}$.*

Proof. The rank 2 case is straightforward, so let us prove the statement for $W = S_n$. Let $r = (ij)$ where $i < j$. Then the assumptions of the lemma are: 1) for any $k \in [1, n]$ one has $w(k) > w(k + 1)$ if and only if $k \in [i, j - 1]$, 2) $j - i = w(i) - w(j)$. It follows that w maps the interval $[i, j]$ to the interval $[w(j), w(i)]$ reversing the order of elements in this interval. One the other hand, w preserves the order of elements in $[1, i]$ and in $[j, n]$. Hence $w(i) = j$, $w(j) = i$ and w stabilizes all elements outside $[i, j]$. □

It follows from this lemma that the map

$$\operatorname{gr}\phi : \operatorname{gr}^F_k(\overline{V}_0) \to \operatorname{gr}^F_k(\overline{V}_1)$$

sends $(v_y, y \in P_i, l(ys_iy^{-1}) = 2k + 1)$ to $(v_w, w \in P_is_i, l(ws_iw^{-1}) = 2k + 1)$ where

$$v_w = (-1)^{|S(ws_iw^{-1})|-1}\tau(w_0(w))v_{w_0(w)w};$$

here $S(ws_iw^{-1})$ is the set of simple reflections in any reduced decomposition of ws_iw^{-1}, and $w_0(w)$ is the longest element in $W_{S(ws_iw^{-1})}$. It follows that $\operatorname{gr}\phi^2$ sends (v_y) to $(\tau(w_0(y))^2v_y)$. By induction it is easy to see that it would be enough to prove the surjectivity of this map for the smallest filtration term $F_l(\overline{V}_0)$ where l is the maximal length of reflections in W.

Assume that W is of classical type. Let r_0 be the reflection corresponding to the maximal positive root. Then r_0 has maximal length in its conjugacy class. We can choose $s_i \in S$ which is conjugate to r_0. Then the space $F_l(\overline{V}_0)$ consists of collections $(v_y, y \in P_i, ys_iy^{-1} = r_0)$ such that $v_y \in V_s$ whenever $sr_0s \neq r_0$, and $v_{sy} - sv_y \in V_s$ if $sr_0s = r_0$.

If $W = S_n$ is the symmetric group, then r_0 is the transposition $(1, n)$. The space $F_l(\overline{V}_0)$ consists of collections $(v_y, y \in P_i, ys_iy^{-1} = r_0)$ such that $v_y \in V_{s_1} \cap V_{s_{n-1}}$ and $v_{s_jy} - s_jv_y \in V_{s_j}$ for $1 < j < n - 1$. Note that the set of $y \in P_i$ such that $ys_iy^{-1} = r_0$ constitutes one coset for the subgroup $W_{[2,n-2]} \subset W$ generated by s_j with $1 < j < n - 1$.

Proof of Theorem 11.6.1. Let $W = S_4$. Then $s_jr_0 \neq r_0s_j$ for $j \neq 2$ and $s_2r_0 = r_0s_2$. We have an involution $(v_y) \mapsto (v_{s_2y})$ on $F_l(\overline{V}_0)$. Consider the corresponding decomposition $F_l(\overline{V}_0) = F^+_l \oplus F^-_l$. Thus, F^+_l consists of collections $(v_y, y \in P_i, ys_iy^{-1} = r_0)$ such that $v_{s_jy} \in V_{s_j}$ for $j \neq 2$, $v_{s_2y} = v_y$, $(s_2 - 1)v_y \in V_{s_2}$. Thus, for any component v_y we have $s_j(v_y) = qv_y$ for $j \neq 2$, $(s_2 - 1)(s_2 - q)v_y = 0$. It is easy to deduce from these conditions that the subrepresentation of B generated by v_y factors through the quadratic Hecke algebra H_q. It remains to apply Theorem 11.5.1. □

11.7. **Some linear algebra.**

Lemma 11.7.1. *Let V be a representation of the cubic Hecke H^c_q corresponding to (W, S) of type B_2, where $q \neq 0$. Assume that $v \in V$ is such that $s_1v = qv$ and $(s_2 - q)(s_2 - 1)v = 0$. Then the subspace $\mathbb{C}[B]v \subset V$ is finite-dimensional.*

Proof. We claim that the subspace V' spanned by elements

$$(x, s_2x, s_1s_2x, s_1^{-1}s_2x, s_2s_1s_2x, s_2s_1^{-1}s_2x, s_1s_2s_1^{-1}s_2x)$$

is closed under s_1 and s_2. To prove this first note that $s_1^{-1}s_2^{-1}s_1^{-1}$ commutes with s_2, and therefore,

$$(s_2 - q)(s_2 - 1)s_1^{-1}s_2^{-1}x = 0.$$

Since $s_1^{-1}s_2x$ is a linear combination of x and $s_1^{-1}s_2^{-1}x$ it follows that

$$(s_2-q)(s_2-1)s_1^{-1}s_2x=0.$$

Similarly since $s_1s_2s_1$ commutes with s_2 we get

$$(s_2-q)(s_2-1)s_1s_2x=0.$$

It follows that the subspace spanned by

$$(x, s_2x, s_1s_2x, s_1^{-1}s_2x, s_2s_1s_2x, s_2s_1^{-1}s_2x)$$

contains $s_2^ns_1^ks_2^l$ for any $n,k,l\in\mathbb{Z}$. It remains to show that $s_2s_1s_2s_1^{-1}s_2x$ and $s_1^2s_2s_1^{-1}s_2x$ belong to V'. Because of the cubic relation we have $s_1^2s_2s_1^{-1}s_2x\equiv s_1^{-1}s_2s_1^{-1}s_2x \mod V'$. Since $s_2s_1^{-1}s_2x$ is a linear combination of $s_1^{-1}s_2x$ and $s_2^{-1}s_1^{-1}s_2x$, it follows that $s_1^{-1}s_2s_1^{-1}s_2x\in V'$. Finally, $s_2s_1s_2s_1^{-1}s_2x=s_1^{-1}s_2s_1s_2^2x$ is a linear combination of $s_1^{-1}s_2s_1s_2x=q^{-1}s_2s_1s_2x$ and $s_1^{-1}s_2s_1x$, so we are done. □

Lemma 11.7.2. *Let s_1, s_2 be a pair of invertible operators on a vector space V such that $(s_1s_2)^2=(s_2s_1)^2$ and $(s_1-q)(s_1^2-1)=0$, and let $v\in V$ be a vector such that $s_1v=qv$, $(s_2-q)(s_2-1)v=0$, and $(s_1s_2)^2v=\mu v$, where $q\neq 0$ and μ are constants. Assume that $q^8\neq 1$. Then $\mu^2\neq 1$.*

Proof. First, we claim that the subspace of V spanned by v and s_2v is closed under s_1 and s_2. Indeed, the identity

$$\mu v=(s_2s_1)^2v=qs_2s_1s_2v$$

implies that

$$s_1s_2v=q^{-1}\mu s_2^{-1}v=q^{-2}(q+1)\mu v-q^{-2}\mu s_2v.$$

Thus, we can assume that V is generated by v and s_2v. If $s_2v=\nu v$, then ν is equal to either 1 or q and $\mu=q^2\nu^2$, so the assertion follows. Otherwise, v and s_2v constitute a basis of V. The matrix of s_1 with respect to this basis is $\begin{pmatrix} q & \frac{\mu(q+1)}{q^2} \\ 0 & -\frac{\mu}{q^2}\end{pmatrix}$. Since s_1 is diagonalizable with eigenvalues among $\{\pm 1, q\}$, it follows that $-\frac{\mu}{q^2}=\pm 1$, i.e. $\mu=\pm q^2$. □

Proof of Theorem 11.6.2. We have $r_0=s_2s_1s_2$, $s_i=s_1$. Thus, the space $F_l(\overline{V}_0)$ consists of pairs $(v_{s_2},v_{s_1s_2})\in(V_{s_2})^{\oplus 2}$ such that $v_{s_1s_2}-s_1v_{s_2}\in V_{s_1}$. We have the natural involution on $F_l(\overline{V}_0)$ interchanging the components, so that $F_l^{\pm}$ consists of $(v,\pm v)$, such that $s_2v=qv$, $(s_1-q)(s_1\mp 1)v=0$. According to Lemma 11.7.1 we can assume that V is finite-dimensional. Also we can assume that V is irreducible as a representation of B. Then the central element $(s_1s_2)^2$ acts as a constant μ on V, and we can apply Lemma 11.7.2 to finish the proof. □

11.8. **Good representations and parabolic induction.** Let $J\subset S$ be a subset, $W_J\subset W$ the corresponding parabolic subgroup, and V_0 a representation of B_J, the braid group corresponding to (W_J,J). In section 10 we associated with V_0 the representation of B in

$$V=\oplus_{x\in W/W_J}V_x,$$

where $V_x=V_0$ such that $b\in B$ sends V_x to $V_{\overline{b}x}$.

Proposition 11.8.1. *If V_0 is a good representation of B_J, then V is a good representation of B.*

Proof. We have a direct sum decomposition

$$K_W(V)\simeq\oplus_{x\in W/W_J}K_{W,x}(V)$$

where

$$K_{W,x}(V)=\{(v_w,w\in W)\mid v_w\in V_{wx}, sv_w-v_{sw}\in(s^2-1)V_{wx}\}.$$

Furthermore, the map $i_y : V \to K_W(V)$ for $y \in W$ decomposes into the direct sum of maps $i_{y,x} : V_{yx} \to K_{W,x}(V)$ where $x \in W/W_J$. Thus, it is sufficient to check that for every $x \in W/W_J$ the images of $i_{y,x}$, $y \in W$ generate $K_{W,x}(V)$. Let $\widetilde{x} \in W$ be a representative of x. Then we have the isomorphism

$$K_{W,x}(V) \widetilde{\to} K_{W,x_0}(V) : (v_w) \mapsto (v_{w\widetilde{x}^{-1}})$$

where $x_0 \in W/W_J$ is the class containing 1. Under this isomorphism the map $i_{y,x}$ corresponds to $i_{y\widetilde{x},x_0}$. Thus, we can assume that $x = x_0$. Now we claim that the canonical projection

$$p_{W_J} : K_{W,x_0}(V) \to K_{W_J}(V_0),$$

leaving only coordinates corresponding to $w \in W_J$, is an isomorphism. Indeed, let $v = (v_w, w \in W) \in K_{W,x_0}(V)$ be an element. For any $x = wx_0 \in W/W_J$ and $s \in S$ such that $sx \neq x$, we have $(s^2 - 1)V_x = 0$, hence $v_{sw} = sv_w$. It follows that for every $w \in W$ we have $v_w = n_{W_J}(w) v_{p_{W_J}(w)}$. Conversely, the latter formula produces an element of $K_{W,x_0}(V)$ from an arbitrary element of $K_{W_J}(V_0)$. Note that for $y \in W_J$, the map i_{y,x_0} corresponds via p_J to the map $i_y : V_0 \to K_{W_J}(V_0)$. Since the B_J-representation V_0 is good, we conclude that K_{W,x_0} is generated by images of i_{y,x_0}, $y \in W_J$. ☐

The main example where the above proposition works is the following. Consider the situation of gluing on the basic affine space of a group G where G and B are defined over the finite field $\mathbb{F}_q$. Then we have the trace map $\mathrm{tr} : K_0(\mathrm{Perv}(G/U)^{\mathrm{Fr}}) \to C(G/U(\mathbb{F}_q))$ where $\mathrm{Perv}(G/U)^{\mathrm{Fr}}$ is the category of perverse Weil sheaves on G/U, and $C(G/U(\mathbb{F}_q))$ is the space of functions $G/U(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l$. The action of the braid group B corresponding to (W, S) on $\mathrm{Perv}(G/U)^{\mathrm{Fr}}$ is compatible with the trace map and induces the action of B on $C(G/U(\mathbb{F}_q))$.

Theorem 11.8.2. *Assume that the center of G is connected. Then the representation of the braid group B on $C(G/U(\mathbb{F}_q))$ is good.*

Proof. We have an action of the finite torus $T(\mathbb{F}_q)$ on $G/U(\mathbb{F}_q)$ by right translation. Let $C_\theta \subset C(G/U(\mathbb{F}_q))$ be the subspace on which $T(\mathbb{F}_q)$ acts through the character $\theta : T(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l^*$. Then we have the direct sum decomposition

$$C(G/U(\mathbb{F}_q)) = \oplus_\theta C_\theta$$

and the action of $b \in B$ sends C_θ to $C_{\overline{b}\theta}$. We claim that if for some character θ and a simple reflection s, one has $s\theta \neq \theta$; then $s^2 = 1$ on C_θ. Indeed, let $f : G/U(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l$ be a function such that $f(xt) = \theta(x) f(x)$. Then

$$\sum_{\lambda \in \mathbb{F}_q^*} f(x\alpha_s(\lambda)) = 0,$$

hence, Proposition 4.1.1 implies that $s^2 f = f$.

Now let O be an orbit of W on the set of characters of $T(\mathbb{F}_q)$. Clearly it is sufficient to prove that the representation of B on $C_O = \oplus_{\theta \in O} C_\theta$ is good. Now we can choose a representative $\theta_0 \in O$ in such a way that all reflections stabilizing θ_0 belong to S. Then the stabilizer of θ_0 in W is the parabolic subgroup $W_J \subset W$ corresponding to some subset $J \subset S$ (this follows from Theorem 5.13 of [10]). As we have shown above the representation of C_O belongs to $\mathrm{Mod}_J -B$; hence it is recovered from the representation of B_J on C_{θ_0} via the construction of section 10. According to Proposition 11.8.1 it is sufficient to check that the representation of B_J on C_θ is good. But on C_θ we have $(s + q^{-1})(s - 1) = 0$ for every $s \in J$ (this follows from the analogue of Proposition 6.2.3 for the finite Fourier transorm); hence we are done by Corollary 11.5.2. ☐

11.9. **Final remarks.** Recall that according to Theorem 5.6.1 and Corollary 6.2.2 we have

$$K_0(\mathcal{A}^{\mathrm{Fr}}) \otimes_{\mathbb{Z}[u,u^{-1}]} \overline{\mathbb{Q}}_l \simeq K_W(K_0(\mathrm{Perv}(G/U)^{\mathrm{Fr}}) \otimes_{\mathbb{Z}[u,u^{-1}]} \overline{\mathbb{Q}}_l)$$

where $\mathcal{A}^{\mathrm{Fr}}$ is the analogue of Weil sheaves in the Kazhdan—Laumon category, and the homomorphism $\mathbb{Z}[u, u^{-1}] \to \overline{\mathbb{Q}}_l$ sends u to q. Hence, we have the natural trace map

$$\mathrm{tr} : K_0(\mathcal{A}^{\mathrm{Fr}}) \to K_W(C(G/U(\mathbb{F}_q))), \tag{11.9.1}$$

induced by the trace maps on every component. If the center of G is connected, then the previous theorem implies the surjectivity of this map. Assuming that the cohomological dimension of $\mathcal{A}$ is finite, Kazhdan and Laumon defined a bilinear pairing on $K_0(\mathcal{A}^{\mathrm{Fr}})$ by taking traces of Frobenius on Ext-spaces. Without this assumption one can still define this pairing for classes of objects of finite projective dimension. It is easy to check that the map (11.9.1) is compatible with this pairing and the non-degenerate form on $K_W(C(G/U(\mathbb{F}_q)))$ defined in Proposition 11.2.3 (where $C(G/U(\mathbb{F}_q))$ is equipped with the standard scalar product in which delta-functions constitute an orthonormal basis). If in addition we know that $K_0(\mathrm{Perv}(G/U)^{\mathrm{Fr}}) \otimes_{\mathbb{Z}[u,u^{-1}]} \overline{\mathbb{Q}}_l$ is a good representation of B, then we can deduce that the map (11.9.1) is the quotient of $K_0(\mathcal{A}^{\mathrm{Fr}})$ by the kernel of the pairing with $K_0(\mathcal{A}^{\mathrm{Fr}})$. Since the representation of the braid group on $K_0(G/U)$ factors through the cubic Hecke algebra, we can check the latter implication in the cases where (W, S) is of type A_2, A_3, or B_2.

On the other hand, using Corollary 9.3.3 we can always define a bilinear pairing on $K_0(\mathcal{A}_m)$ with values in the field of Laurent series $\overline{\mathbb{Q}}_l((u))$ by looking at the action of Frobenius on weight-n components of the Ext-spaces. In the cases when the representation of the braid group on $K_0(\mathrm{Perv}_m((G/U)_0)) \otimes_{\mathbb{Z}[u,u^{-1}]} \mathbb{Q}(u)$ is good, the Laurent series obtained as values of this pairing are actually rational. We conjecture that in fact they are always rational. Furthermore, a similar pairing can be defined in the case when the Frobenius automorphism is twisted by the action of some $w \in W$. We conjecture that it still takes values in rational Laurent series and that the quotient of $K_0(\mathcal{A}_m^{w\,\mathrm{Fr}}) \otimes_{\mathbb{Z}[u,u^{-1}]} \overline{\mathbb{Q}}_l((u))$ by the kernel of this pairing is finite-dimensional.

12. Appendix. Counterexample to the conjecture of Kazhdan and Laumon.

In this appendix we will show that the Kazhdan—Laumon category $\mathcal{A}$ has infinite cohomological dimension in the simplest non-trivial case $G = \mathrm{SL}_3$.

For every $w \in W$ let us denote by O_w the simple object in $\mathcal{A}$, such that the corresponding gluing data is the constant sheaf $\overline{\mathbb{Q}}_{l,X}[d]$ at the place $w \in W$, and zero at all the others, where $d = \dim X$. Recall that for every $w \in W$ the functor $j_{w,!} : \mathrm{Perv}(X) \to \mathcal{A}$ has the left derived one $Lj_{w,!}$ (see Proposition 7.1.2 and the remark after it). So for every $w \in W$ we can introduce the following object in the derived category of $\mathcal{A}$: $P_w = Lj_{w,!}(\overline{\mathbb{Q}}_{l,X}[d])$. Note that by Proposition 7.1.2 the functor $Lj_{w,!}$ is left adjoint to the corresponding restriction functor, hence, we have

$$V^*_{w,w'} := \mathrm{Ext}^*(P_w, O_{w'}) = \begin{cases} 0, \ w \neq w', \\ H^*(X) = H^*(G), w = w'. \end{cases}$$

(here cohomology is taken with coefficients in $\overline{\mathbb{Q}}_l$). We want to compare these spaces with the spaces

$$E^*_{w,w'} := \mathrm{Ext}^*(O_w, O_{w'})$$

(in fact, $E^*_{w,w'}$ depends only on $w'w^{-1}$). This is done with the help of the following lemma.

Lemma 12.0.1. *For every i we have a canonical isomorphism in $\mathcal{A}$*

$$H^{-i}(P_w) = \oplus_{w', \ell(w')=i} O_{w'w}(i).$$

Remark. Most of the results in this paper can be translated into the parallel setting where algebraic D-modules on the complex algebraic variety $(G/U)_{\mathbb{C}}$ are used instead of l-adic sheaves over the corresponding variety in characteristic p. The main result of [5] asserts that the corresponding "glued" category is equivalent to the category of modules over the ring of global differential operators on $(G/U)_{\mathbb{C}}$. The functor Rj_* (the Verdier dual to the D-module counterpart of the functor $Lj_!$ considered above; it is

somewhat more natural to work with this Verdier dual functor in the D-module setting) is then identified with the derived functor of global sections from the category of D-modules to the category of modules over the global differential operators.

The D-module counterpart of Lemma 12.0.1 is easily seen to be equivalent to the Borel-Weil-Bott Theorem (which computes cohomology of an equivariant line bundle on the flag variety $(G/B)_{\mathbb{C}}$).

Proof of Lemma 12.0.1. Recall that by Proposition 7.1.2 the composition of functors $j^*_{w'w} \circ Lj_{w,!}$ coincides with the left derived functor of ${}^pH^0\, F_{w',!}$. Furthermore, Theorem 8.1.1 implies that under the identification of the derived category of $\mathrm{Perv}(X)$ with $D^b_c(X, \overline{\mathbb{Q}}_l)$ the latter derived functor coincides with $F_{w',!}$. Therefore, we have

$$j^*_{w'w}P_w = j^*_{w'w}Lj_{w,!}(\overline{\mathbb{Q}}_{l,X}[d]) \simeq F_{w',!}(\overline{\mathbb{Q}}_{l,X}[d]).$$

According to Lemma 6.1.1 for every simple reflection s we have

$$F_{s,!}(\overline{\mathbb{Q}}_{l,X}[d]) = (\overline{\mathbb{Q}}_{l,X}[d])[1](1).$$

Therefore, for every $w' \in W$ we have

$$F_{w',!}(\overline{\mathbb{Q}}_{l,X}[d]) = (\overline{\mathbb{Q}}_{l,X}[d])[\ell(w')](\ell(w')).$$

Hence, for every i we have

$$j^*_{w'w}H^{-i}(P_w) = \begin{cases} 0, \ \ell(w') \neq i, \\ \overline{\mathbb{Q}}_{l,X}[d](i), \ \ell(w') = i. \end{cases}$$

This immediately implies our statement. ∎

Thus, we have a spectral sequence with the E_2-term

$$\oplus_{p\leq 0, q\geq 0} \oplus_{\ell(w_1)=-p}\, E^q_{w_1w,w'}(p)$$

converging to $V^*_{w,w'}$. Now let us assume that the spaces $E^*_{w,w'}$ are finite-dimensional and lead this to contradiction. Note that all our spaces carry a canonical (mixed) action of Frobenius which is respected by this spectral sequence. We can encode some information about Frobenius action on such a space by considering Laurent polynomials in u, where the coefficient with u^n is the super-dimension of the weight-n component. Let $e_{w,w'}$ (resp. $v_{w,w'}$) be such a Laurent polynomial in u corresponding to $E^*_{w,w'}$ (resp. $V^*_{w,w'}$). Then the above spectral sequence implies that

$$v_{w,w'} = \sum_{w_1\in W} e_{w_1w,w'}(-u)^{\ell(w_1)} \tag{12.0.2}$$

Now recall that $v_{w,w'} = 0$ for $w \neq w'$ while $v_{w,w}$ is equal to the Poincare polynomial of G $p_G(u) = \prod(1-u^{e_i})$, with e_i running over the set of exponents of G. Therefore, we can rewrite (12.0.2) in the matrix form as follows. Let us consider the matrix $E = (e_{w,w'})$ with rows and columns numbered by W and with entries in $\mathbb{Z}[u,u^{-1}]$. Let us also define the matrix M of similar type by setting

$$M_{w,w'} = (-u)^{\ell(w'w^{-1})}.$$

Then (12.0.2) is equivalent to the following equality of matrices:

$$ME = p_G \cdot I \tag{12.0.3}$$

where I is the identity matrix.

Now we are going to show that M does not divide $p_G \cdot I$ for $G = \mathrm{SL}(3)$ in $Mat_6[u,u^{-1}]$ (as $\det(M)$ vanishes at a primitive 6-th root of 1, while p_G does not).

Recall that for any finite group G one can form a matrix $M^G \in Mat_n(\mathbb{Z}[x_1,..,x_n])$, $n = |G|$; here rows/columns of M^G and variables x of the polynomial ring are indexed by elements of G, and we set $M^G_{g_1,g_2} = x_{g_1g_2}$. It is well known that $\det(M^G)$ is the product of factors indexed by irreducible representations of G, and the degree of a factor, as well as the power in which it enters the decomposition

equals the dimension of the representation. Moreover, the factor corresponding to a (1-dimensional) character of G is $\sum \chi(g) x_g$.

Now our matrix M is obtained from M^G for $G = W$ by $x_w \mapsto (-u)^{\ell(w)}$ and multiplication by a matrix of the permutation $w \mapsto w^{-1}$. So $\det(M)$ is divisible by $(\sum_{w\in W} u^{\ell(w)}) \cdot (\sum_{w\in W} (-u)^{\ell(w)})$. For $G = \mathrm{SL}(3)$ we get that $\det(M)$ is divisible by $1+2u+2u^2+u^3$ and $1-2u+2u^2-u^3$. The latter polynomial vanishes at a primitive 6-th root of 1. However, $p_{\mathrm{SL}_3} = (1-u^2)(1-u^3)$, hence the equality (12.0.3) is impossible.

Note that this counterexample does not contradict to the results of section 11 since in that section we considered the localization of $K_0(\mathcal{A})$ on which u does not act as a root of unity.

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